The Logic of Problem Generation: from Morality and Solving to De-Posing and Rebellion*

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I Some personal ruminations

As Peter Hilton indicated in his address two days ago, this group is testimony to the fact that though endangered, "small" is not extinct. It also however is testimony to something more precious—to the fact that small is not incompatible with diversity of point of view, and more importantly with mutual respect for that diversity. What has been most refreshing to discover is that there is greater *within* than *between* group diversity among mathematicians, mathematics educators, and school teachers.

I suspect that this address will reveal yet another kind of diversity and even incompatibility for which there may be slightly less tolerance: namely *within* person inconsistencies. The problem (to use a word that will invite you to view what I say recursively) is that I have thought about the subject matter of this talk for a long time. As a matter of fact, the first article in which my colleague, Marion Walter, and I ventured into the territory was published by David Wheeler when he was editing *Mathematics Teaching*. [Walter and Brown, 1969] Furthermore, not only have I recently published an article with the same theme in *FLM* [Brown, 1981], but our thinking is about to culminate in a book that draws together a decade and a half's worth of playing around with the idea of problem generation [Brown and Walter, 1983].

I view my problem today as one of providing some novelty such that I will not be bored by yet one more foray into the field. I could of course add a small niche to that already established tradition. After considerable reflection however, I have decided to try something more personally challenging. I will reconstruct for you a large portion of the entire terrain, but I will attempt to do so through a new set of lenses. Though I will occasionally reproduce category distinctions and examples I have previously devised, I will be approaching much of what looks like repetition from a new enough perspective so that you will have the opportunity to help unearth for me not only new potential, but the existence of inconsistencies I have alluded to earlier. For those of you who are familiar with what I have previously written and who wish to get on with the mystery, I recommend that you focus upon my comments dealing with morality and with the relationship of a problem to a situation.

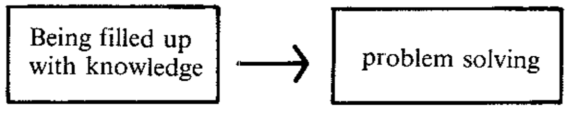
II The rhetoric of problem solving

At the end of his presentation, Peter Hilton made a couple of remarks about problem solving that provide a natural entrée for much of what I want to say. [The presentation referred to is printed in this issue. Although Peter Hilton's remarks were by way of an aside, and are not recorded in his paper, I have retained Stephen Brown's reference since the differences in their viewpoints seem important. — Ed.]

To begin, he points out correctly that one does not merely solve problems in the abstract; rather one solves specific problems. His point then is that one has to know things (and preferably a lot of things) before one can solve problems, and that it is a mistake to engage people in problem solving behavior before they have acquired some healthy repertoire of knowledge. The implication is that we might be better advised to familiarize students with a substantial amount of mathematics before we engage them in solving problems.

Now, it certainly is true that one does not solve problems "in general." If I ask you what problem you are trying to solve, and you respond with, "I'm not trying to solve any specific problem at all. I'm just solving problems in general," I might have good reason to doubt not that you may be a good problem solver, but rather that you understand the *meaning* of problem solving in the first place.

But to say that one solves problems by working on specific "things," does not imply that those "things" themselves are "acquired" totally independently of problem solving. That is, the picture Peter Hilton has conveyed is that the following are logically or temporally related from left to right.



While it may be true that a "thing" (call it knowledge if you wish) is needed as a prerequisite for solving problems, it is I believe a fundamental pedagogical error to act as if those "things" can ever be acquired much as an empty vessel can be filled up "Coming to know" anything is radically different from being filled up and the former shares some important elements with the activity of problem solving.

I find Peter Hilton's other comment with regard to problem solving and the curriculum more compatible with much of what I believe to be the case. He comments that a major difficulty with the present educational interest in problem solving is that it focuses attention at the wrong spot with regard to inquiry That is, it leads us to focus on answers or solutions rather than upon questions. Though he may be correct in terms of existing practice, and he does capture an important truth, the situation in terms of the logic and the potential practice of problem solving in the curriculum requires considerably more "unpacking" than that brief remark would seem to warrant. Much of what follows will be an effort to explore the nature of the interrelationship and the independence of problem solving and problem generating

In preparation for providing such linkages, I would like to dwell a little longer on problem solving *per se*. By the end of this section, the compelling need to relate the two will begin to emerge.

Two cultures

A quarter of a century ago, C. P. Snow accurately pointed out how little the two cultures—roughly the sciences and the humanities—have learned to understand each other and to gain from the wisdom they each have to offer. [Snow, 1959]

> Between the two a gulf of mutual incomprehension —sometimes . . hostility and dislike, but most of all lack of understanding [emerges]. They have a curious distorted image of each other . . . non-scientists tend to think of scientists as brash and boastful . . [They] have a rooted impression that the scientists are shallowly optimistic, unaware of man's condition On the other hand, the scientists believe that the literary intellectuals are totally lacking in foresight, peculiarly unconcerned with their brother men, in a deep sense anti-intellectual, anxious to restrict both art and thought to the existential moment. [p. 12]

Not only are their problem solving styles different, but more importantly there are divergent views on what it means for something to be solved. It is worth observing that as a profession, mathematics education is almost by definition bound to the schizophrenic state of searching for and creating the "snow-capped" bridges; for mathematics is more closely aligned with the culture and world view of science and education than with the humanities.

As we search for a better understanding of what problem solving might be about, however, we have not only neglected to build bridges, but we have tended to ignore most non-mathematical educational terrain that might be worth connecting in the first place.

In particular, we have overlooked those educational efforts in other fields which have been concerned with problem solving but have indicated that concern through a different language. Dewey's analysis of "reflective thought" and of the concept of "intelligence" would seem to offer a rich complement to much of the problem solving rhetoric The role of *doubt, surprise* and *habit* in problem solving explored by Dewey would seem to complement much of the influential work of Polya, and would offer options we have not yet incorporated in much of our thinking about problem solving in the curriculum [Dewey, 1920, 1933]

We have much to learn about the role of dialogue in problem solving, something we in mathematics education have tended to view in pale "discovery exercise" terms at best. Yet the use and analysis of dialogue in educational settings has been the hallmark not only of English education, but of several curriculum programs in other fields as well. "Public controversy" in the social studies in the late 60's and early 70's was a central theme around which students were taught not only to carry on intelligent dialogue, but more importantly to unearth and to discuss controversial and sometimes incompatible points of view. [Oliver and Newman, 1970] It would enrich considerably what it is we call problem solving in mathematics, if we were to entertain the possibility that for logical as well as pedagogical reasons, we might encourage not merely complementary, but incompatible perspectives on a problem or a series of problems Furthermore such curriculum in the social studies as well as in the newly emerging field of philosophy for children might enable us to help students *appreciate* irreconcilable differences rather than to resolve or dissolve them as we are prone to do in mathematics [Lipman, *et al.*, 1977]

"Critical" thinking is another "near relative" of problem solving that began influencing the curriculum in schools as far back as the progressive education era, and there is a considerable history of efforts to integrate different disciplines through the use of critical thought. [Taba, 1950] It is a history that is worth understanding not only because of its connection with problem solving, but because the theme is presently undergoing rejuvenation in the non-scientific disciplines much as problem solving has re-emerged in mathematics and science

Moral education and Kohlberg

In closing this section, we turn towards one area within which the tunes of critical thinking have been re-sung recently—that of moral education. The issues that emerge here and those that we develop in the next section are part of the new (and not yet well integrated) backdrop mentioned in the first section.

First of all, we might ask why critical thinking and moral education have been joined at all. To many people, they would seem to occupy different poles. The connection hinges on our concern for the teaching of values in a pluralistic, democratic society. How do we go about such education in a public school setting without indoctrinating with regard to a particular religious or ethnic point of view? Though we might argue over whether or not *it* is a set of values itself and if so, why it is that such a collection is more neutral than any religious or ethnic point of view, the liberal tradition of thinking critically about whatever values one adopts does provide an entrée for those concerned with morality in a pluralistic society.

Though there a number of different kinds of programs within which moral education is taught [Lickona, 1976], most of them rely heavily upon contrived or natural dilemmas as a starting point. Our focus here will be on Kohlberg's program of moral development and education. A typical dilemma he has used for much of his research and for his deliberate program of education as well is the Heinz dilemma:

> In Europe, a woman was near death from a rare form of cancer. There was one drug that the doctors thought might save her, a form of radium that a druggist in the same town had recently discovered. The druggist was charging $2000, ten times what the drug cost him to make. The sick woman's husband, Heinz, went to everyone he knew to borrow the money, but he could only get together about half of what the drug cost. He told the druggist that his wife was dying and asked him to sell it cheaper or let him pay later. But the druggist said, "no." So Heinz got desperate and broke into the man's store to steal the drug for his wife. [Kohlberg, 1976, p. 42]

Should Heinz have stolen the drug? Based upon an analysis of longitudinal case studies to answers of dilemmas of this sort, Kohlberg has created a scheme of moral growth that he claims is developmental. Furthermore, he has created not only a research tool but an educational program around such dilemmas. It is through discussing and justifying responses to such dilemmas that students mature in their ability to find good reasons for their choices.

It is not the specific value that one chooses (e.g., steal the drug vs. allow the wife to die), but the reasons offered for the decision that places people along a scale of moral development.

At the lowest level of moral maturity, (pre-conventional) Kohlberg finds that people argue primarily from an awareness of punishment and reward. Thus someone at a lowest stage of development might claim that Heinz should not steal the drug because he would be punished by being sent to jail, or he might claim that he should steal it because his wife might pay him well for doing so. It is almost as if the punishment inheres in the action itself. At a later stage (conventional) people argue from the more abstract perspective of what is expected of you and also from the point of view of the need to maintain law and order. At the highest stage of principled morality, one argues on the basis not of rules that could conceivably change but with regard for abstract principles of justice and respect for the dignity of human beings. Such principles single out fairness and impartiality as part of the very definition of morality.

None of these structural arguments (e.g., punishment/reward, law and order, justice) in themselves dictate what is a correct resolution of any dilemma. Rather they form part of the web that is used to justify the decisions made, and it is in listening to these reasons that Kohlberg and his followers are capable of deciding upon one's level of moral development.

Gilligan's challenge

Despite the fact that Kohlberg's scheme for negotiating moral development neglects to focus upon action, it is a refreshing counterpoint to a program of moral education which conceives of its role as one of inculcating specific values in the absence of reason. Nevertheless, there has been some penetrating criticism of his scheme recently—a criticism which condemns much of Kohlberg's work on grounds of sexism. That is, Kohlberg's research and ultimately his scheme for what represents a correct hierarchy of development is based upon his longitudinal research *only with males*. Once the scheme was created and the stages developmentally construed, Kohlberg interviewed females and concluded that their deviation from the established hierarchical scheme implied an arrested form of moral development.

Gilligan [1982] points out that the existence of a totally different category scheme for men and women not only may be a consequence of different psychological dynamics, but rather than exhibiting a logically inferior mind set, it suggests moral categories that are desperately in need of incorporation with those already derived. Compare the following two responses to the Heinz dilemma, one by Jake, an eleven-year old boy and the second by Amy, an eleven-year old girl. Jake is clear that Heinz should steal the drug at the outset, and justifies his choice as follows:

> For one thing a human life is worth more than money, and if the druggist makes only $1000, he is still going to live, but if Heinz doesn't steal the drug, his wife is going to die. (Why is life worth more than money?) Because the druggist can get a thousand dollars later from rich people with cancer, but Heinz can't get his wife again. (Why not?) Because people are all different and so you couldn't get Heinz's wife again. [Gilligan, 1982, p. 26]

Amy on the other hand equivocates in responding to whether or not Heinz should steal the drug:

> Well, I don't think so. I think there might be other ways besides stealing it, like if he could borrow the money or make a loan or something, but he really shouldn't steal the drug—but his wife shouldn't die either. If he stole the drug, he might save his wife then, but if he did, he might have to go to jail, and then his wife might get sicker again, and he couldn't get more of the drug, and it might not be good. So, they should really just talk it out and find some other way to make the money. [p. 28]

Notice that Jake *accepts* the dilemma and begins to argue over the relationship of property to life. Amy, on the other hand, is less interested in property and focuses more on the interpersonal dynamics among the characters. More importantly, Amy refuses to accept the dilemma as it is stated, but is searching for some less polarized and less of a zero sum game.

Kohlberg's interpretation of such a response would imply that Amy does not have a mature understanding of the nature of the moral issue involved—that she neglects to appreciate that this hypothetical case is attempting to test the sense in which the subject appreciates that in a moral scheme life takes precedence over property. Gilligan on the other hand in analyzing a large number of such responses has concluded not that the females are arrested in their ability to move through his developmental scheme, but that they tend to abide by a system which is orthogonal to that developed by Kohlberg—a system within which the concepts of *caring* and *responsibility* rather than *justice* and *rights* ripen over time.

Gilligan [1982] comments with regard to Amy's response:

> Her world is a world of relationships and psychological truths where an awareness of the connection between people gives rise to a recognition of responsibility for one another, a perception of the need for response. Seen in this light, her understanding of morality as arising from the recognition of relationship, her belief in communication as the mode of conflict resolution, and her conviction that the solution of the dilemma will follow from its compelling representation seem far from naive or cognitively immature [p 30]

The difference between a "Kohlbergian" and a "Gilliganish" conception of morality is well captured by two different adult responses to the question, "what does morality mean to you?" [Lyons, 1983] A man interviewed comments:

> Morality is basically having a reason for doing what's right, what one ought to do; and, when you are put in a situation where you have to choose from amongst alternatives, being able to recognize when there is an issue of "ought" at stake and when there is not; and then having some reason for choosing among alternatives. [p. 125]

A woman interviewed on the same question comments:

> Morality is a type of consciousness, I guess a sensitivity to humanity, that you can affect someone else's life. You can affect you own life and you have the responsibility not to endanger other people's lives or to hurt other people. So morality is complex. Morality is realizing that there is a play between self and others and that you are going to have to take responsibility for both of them It's sort of a consciousness of your influence over what's going on. [p. 125]

While Gilligan and her associates do not claim that development is sex bound in such a way that the two systems are tightly partitioned according to gender, they do claim to have located a scheme that tends to be associated more readily with a female than a male voice. Behind the female voice of responsibility and caring, some of the following characteristics appear to me to surface:

1. A context-boundedness,
2. A disinclination to set general principles to be used in future cases,
3. A concern with connectedness among people.

Though not all of these characteristics are exhibited in Amy's response, they do appear in interviews with mature women. *Context-boundedness* represents a plea for more information that takes the form not only of requesting more details (e g., what is the relationship between husband and wife?) but of searching for a way of locating the episode within a broader context. Thus unlike men, mature women might tend to respond not by trying to resolve the dilemma, but by exhibiting a sense of *indignation* that such a situation as the Heinz dilemma might arise in the first place. Such a response might take the following form: "The question you should be asking me is 'What are the horrendous circumstances that caused our society to evolve in such a way that dilemmas of this sort could even arise—that people have learned to miscommunicate so poorly'?"

The second characteristic I have isolated above, is an effort to attempt to understand each situation in a fresh light, rather than in a legalistic way—i e., in terms of already established precedent. Connected with context boundedness it is the desire to see the fullness of "this" situation in order to see how it might be *different from* (and thus require new insight) rather than compatible with one that has already been settled.

With regard to the third characteristic, conflict is less a logical puzzle to be resolved but rather an indication of an unfortunate fracture in human relationships—something to be "mended" rather than an invitation for some judgement.

In the next section we turn towards a consideration, in a rather global way, of how it is that a Gilliganish perspective of morality might impinge on the study of mathematics. While we have not yet drawn any explicit links, it is not difficult to intuit not only that it threatens the status quo but that it sets a possible foundation for the relationship of problem generation to problem solving. Though we shall focus upon the findings from the field of moral education, we do not wish to lose sight of some of the other humanistic areas of curriculum from which mathematics education might derive enlightenment.

III Kohlberg vs. Gilligan: The transition from solving to posing

It surely appears that problem solving in mathematics education has been dominated by a Kohlbergian rather than a Gilliganish one. Gilligan herself has an intuition for such a proposition, when she comments with regard to Jake's response to the Heinz dilemma:

> Fascinated by the power of logic, this eleven-year old boy locates truth in math, which he says is "the only thing that is totally logical." Considering the moral dilemma to be "sort of like a math problem with humans," he sets it up as an equation and proceeds to work out the solution. Since his solution is rationally derived, he assumes that anyone following reason would arrive at the same conclusion and thus that a judge would consider stealing to be the right thing for Heinz to do. [p. 26-27]

The set of problems to be solved as well as the axioms and defintions to be woven into proofs are part of "the given"—the taken-for-granted reality upon which students are to operate. It is not only that the curriculum is "depeopled" in that contexts and concepts are for the most part presented ahistorically and unproblematically, but as it is presently constituted the curriculum offers little encouragment for students to move beyond merely accepting the non-purposeful tasks

Furthermore, rather than being encouraged to try to capture what may be *unique* and unrelated to previous established precedent in a given mathematical activity (the legalistic mode of thought we referred to as the second characteristic behind Gilligan's analysis of morality as responsibility and caring), much of the curriculum is presented as an "unfolding" so that one is "supposed" to see similarity rather than difference with past experience It is commonplace surely in word problems to tell people to *ignore* rather than to embellish matters of detail on the ground that one is after the underlying structure and not the "noise" that inheres in the problem.

In so focusing on essential isomorphic features of structures, the curriculum tends not only to threaten a Gilliganish perspective, but as importantly, it supports only one half of what I perceive much of mathematics to be about. That is, mathematics not only is a search for what is essentially common among ostensibly different structures, but is as much an effort to reveal essential differences among structures that appear to be similar. [See Brown, 1982a]

With regard to context boundedness, there is essentially no curriculum that would encourage students to explicitly ask questions like:

> What purpose is served by my solving this problem or this set of problems?
>
> Why am I being asked to engage in this activity at this time?
>
> What am I finding out about myself and others as a result of participating in this task?
>
> How is the relationship of mathematics to society and culture illuminated by my studying how I or other people in the history of the discipline have viewed this phenomenon?

Elsewhere [Brown 1973, 1982] I have discussed how I first began to incorporate such reflection as part of my own mathematics teaching, and presently I shall have other illustrations. There are a number of serious questions that must be thought through, however, before one feels comfortable in encouraging the generation and reflection of the kinds of questions indicated above We need to be asking ourselves whether or not that kind of reflection represents respectable mathematical thinking. In addition we ought to be concerned about the ability of students to handle that thinking in their early stages of mathematical development.

It is interesting to observe that though we are cajoled by many to integrate mathematics with other fields, the "real world" applications seem to be narrowly defined in terms of the scientific rather than the humanistic disciplines. In particular questions of value or ethics are essentially nonexistent. That is particularly surprising in light of the fact that a major rationale for relating mathematics to other fields seems to be that such activity may enable students to better solve "real world" problems that they encounter on their own. I know of essentially no "real world" problems that one decides to engage in for which there is not embedded some value implications.

McGinty and Meyerson [1980] suggest some steps one might want to take to develop curriculum for which value judgements are an explicit component. Consider a problem like the following:

> Suppose a bag of grass seed covers 400 square feet. How many bags would be needed to uniformly cover 1850 square feet? [p 501]

So far so dull It is not only that for many students the above would not constitute a problem, but more importantly it lacks any reasonable conception of context-boundedness. The authors, however, go on to suggest inquiry that is more "real worldish" that most of the word problems students encounter. They ask:

> Should the person buy 5 bags and save the leftover—figuring prices will rise next year? Buy 5 bags and spread it thicker? Buy 4 bags and spread it thinner? [p 502]

Once we become aware of ethical/value questions as a central component of decision making, it is clear that there is much more we might do in the way of generating problems for students as well as encouraging them to do so on their own. One of the *au courant* curriculum areas is probability and statistics As a profession, we correctly appreciate that we need to do more to prepare students to operate in an uncertain world, wherein one's fate is not sown with the kind of exactitude that much of the earlier curriculum has implied In creating such a curriculum, however, we continue to give the false illusion that mathematical competence is all that is required to decide wisely. Compare *any* probability problem (selected at random of course) from any curriculum in mathematics with the following probability problem:

> A close relative of yours has been hit by an automobile. He has been unconscious for one month. The doctors have told you that unless he is operated upon, he will live but remain a vegetable for the rest of his life They can perform an operation which, if successful, would restore his consciousness. They have determined, however, that the probability of being successful is .05, and if they fail in their effort to restore consciousness, he will certainly die.

What counsel would you give the doctors? One could clearly embed the above problem in a more challenging mathematical setting, for example, setting up the conditions that would have enabled one to arrive at the .05 probability (or perhaps modifying it so that outer limits are set on the probability of survival) but nevertheless, it is such ethical questions in many different forms that plague most thinking people as they go through life making decisions.

Is such a problem generation on the part of the teacher or student an ingredient of mathematical thought? I do not think the answer is clear. There is nothing god-given and written in stone that establishes what is and is not part of the domain of mathematics, and clearly what has constituted legitimate thinking in the discipline has changed considerably over time. Even if questions of the kind we have been raising in this section, however, would move us in directions that are at odds with the dominant and respectable mode of mathematical thought, it is worth appreciating that as educators we have a responsibility to future citizens that transcends our passing along *only* mathematical thought. The latter appears to me to be a very narrow view of what it means to educate. In realizing that only a very small percentage of our students will be mathematicians, we have not adequately explored our obligation to those who will not expand the field *per se*. We have mistakenly identified our task for the majority as one of "softening" an otherwise rigorous curriculum. What may be called for is an ever more intellectually demanding curriculum, but one in which mathematics is embedded in a web of concerns that are more "real world" oriented than any of us have begun to imagine.

Is it worth observing that such complication of mathematical thinking may in fact pose a major threat to a concept that we have begun in recent years to revere—that of mathematization. In attempting to find reason to believe that children can indeed function as mathematicians (as opposed to exhibiting routine imitative skills), David Wheeler [1982] looks towards exceptional cases of mathematical precocity. He comments:

> I don't see children however exceptional functioning as historians, or a lawyers, or as psychologists, for instance, since these are extremely complex functionings that involve subtle relationships between (sic) several frames of reference. But I would hypothesize that mathematics belongs with art, music, writing and possibly science, as one of a class of activities that require only a particular kind of response to be made by an individual to his immediate, direct experience. [p. 45]

While I would certainly not wish to pit mathematization, as Wheeler describes it, against the mindless symbol-pushing that represents its polar opposite, I believe that as educators we are obliged to push the bounds of complicating that discipline in an effort to engage the minds of students in directions that define their humanity.

IV Down from a crescendo

How do we descend from the heights and perhaps the overinflated language which concluded the previous section? Perhaps one way is to take stock of where we have been led and to try to sharpen the implication that might follow. The confrontation between Kohlberg and Gilligan has served two purposes that appear on the surface to be very different. First of all, we have used the challenge of Gilligan's research to point out that there is a world view that has achieved empirical expression with regard to issues of morality but which is worth taking seriously in other domains as well. Moving beneath the concepts of caring and responsibility established by Gilligan, we find dimensions that are not strictly moral in character but which deal with *purpose*, *situation specificity* (a non-legalistic mode) and *people connectedness*. We have suggested that very little of the existing mathematics curriculum caters to those characteristics, and in fact the dominant mode caters to their opposite.

Secondly, we have not only used Gilligan in contrast to Kohlberg to establish broad categories within which the present curriculum is deficient, but we have pointed out that what the two perspectives have in common—namely a concern with morality—represents a field of inquiry that may be as important to integrate with mathematical thinking as are the more standard disciplines that form the backbone of more conventional applications.

Both of these perspectives have potentially revolutionary implications. They not only suggest the need for both teacher and student to incorporate a more serious problem generating perspective (including the broad types of questions raised at the beginning of the previous section) as an essential ingredient of problem solving, but they have the potential to infect every aspect of mathematics education from drill and practice, to an understanding of underlying mathematical structures.

Our goal for the remainder of this paper will be the more modest one of making a case for the inclusion of problem generating strategies within the curriculum. I will for the most part be drawing upon and integrating ideas that I have previously developed. While I will make minimal explicit reference to the Gilligan perspective, I believe it is possible to view much of what follows as being derived from what I have referred to as the underlying components of caring and being responsible. The joining of links explicitly in other mathematics education areas is a task to be left for another time (and perhaps another person).

V Our knee-jerk solving mentality

Any field of inquiry establishes a common language among its investigators. The same kind of phenomenon is exhibited among friends, lovers and members of a family. It is frequently possible to determine the extent to which you are in fact an "outsider" by the degree to which you are incapable of understanding the short-circuiting of language among participants. There is certainly good reason for members of an "in-group" to engage in such short-circuiting behavior. In addition to merely increasing efficiency of communication, there are important psychological and sociological bonds established through such behavior.

Nevertheless, we sometimes pay a price for the common language we establish. That is, in focusing on common understanding, we not ony leave out other perspectives, but we may be *unaware* of what we are leaving out. The specialization that results from such behavior not only may leave us unaware of what we have left out, but worse than that, we may even lose our ability to incorporate those awarenesses within our world view even when they are pointed out to us.

As educators, it is worth taking stock every so often to examine explicitly what we are leaving out in the common language we are establishing with our students. Such an instance occurred a number of years ago at which point Marion Walter and I were team teaching a course on problem solving. We were doing work in number theory, and were hoping to derive a formula to generate primitive Pythagorean triplets. We began the lesson by asking:

$x^2 + y^2 = z^2$. What are some answers?

Responses began to flow, and students responded with:

3, 4, 5
5, 12, 13
8, 15, 17

After a while, a smile broke out on the face of a student who responded:

1, 1, $\sqrt{2}$

A few more "courageous" and humorous responses then were suggested like:

$-1, -1, \sqrt{2}$

Marion and I then jokingly reprimanded the "deviants," and proceeded to explore what we were about in the first place—a search for a generating formula After class, however, we began to talk to each other about the incident It hit us very hard that the "deviants" were beginning to appreciate something that has occupied a considerable part of our collective energy for the past fifteen years. What struck us was that:

$x^2 + y^2 = z^2$. What are some answers?

has a kind of foolishness about it that derives from the closed position of a common language with its unspoken but built-in assumptions. Notice that $x^2 + y^2 = z^2$ is not even a question. How can one come up with answers?

Yet the students dutifully did come up with answers, because they carried along a host of assumptions that we in fact have trained (implicitly) them to accept. They assumed (at least at the beginning) that the symbolism had connoted that the domain was natural numbers. Furthermore they assumed that the symbolism was calling for something algebraic; and within that context they assumed that we were searching for instances that would make an open sentence true

As soon as we began to appreciate that "the deviants" had begun to appreciate something we had not seen, we realized that there was a whole new ball game at stake We had not realized at the time that in expanding this concept for this class, we were opening Pandora's box.

After realizing that we had implicitly assumed that the domain was natural numbers, we encouraged students to ask such new questions as:

> For what *rational* numbers x, y, z is it true that $x^2 + y^2 = z^2$?

Realizing that we had implicitly assumed that we were searching for true instances of the open sentence, we encouraged students to ask such new questions as:

> For what natural numbers is it true that $x^2 + y^2 = z^2$ is "almost" true? (e.g., 4, 7, 8 misses the equality by 1).

Realizing that we had implicitly assumed that the question was algebraic, the students began to ask a host of geometric questions that derived from connotations of the algebraic form.

What followed immediately was one of the most intellectually stimulating units that either of us had previously experienced with our students, and what dawned eventually on all of us was something that has had a lasting effect

First of all, we began to appreciate that such deviations from standard curriculum are not mere frills. That is, in exploring such questions as the "almost" primitive Pythagorean triplet question, all of us gained a much clearer understanding of what the actual primitive Pythagorean triplet question was in fact about—not only from the point of view of statement but of proof as well.

Secondly, and more importantly, we began to realize that an implicit part of the common language we share with students is one which focuses upon and points so strongly towards the search for solutions and answers, that we continue to search for answers even when no question is asked at all! We were thus launched on our journey to try to understand the role of problem generation in the doing of mathematics.

VI Posing and deposing: a first step

I am beginning to appreciate an important aspect of what is behind an understanding of the role of posing problems that I have not seen before, despite the fact that I have referred in much of my writing to examples within which this issue is embedded. For a number of years, educators have appreciated that there might be considerable value in giving students not *problems* to solve but *situations* to investigate. Higginson [1973], for example, locates a number of characteristics of what he refers to as "potentially rich situations." Situations are much "looser" than problems, and situations themselves do not ask built-in questions It is the job of the student to create a question or pose a problem. Geoboards, Cuisenaire Rods, polyominoes are all examples of situations, but situations need not be concrete materials; they can be abstractions as well.

What I have recently (in preparing for this talk) began to appreciate is that the pedagogical issue is much deeper and more interesting than that of directing teachers to create rich situations for their students to investigate The issue is even more complicated than providing both mechanisms and an atmosphere within which problem solving might be isolated from situation-related activities. Rather the pedagogical task is one of enabling all of us to appreciate the differences between a problem and a situation, and of finding ways to move from one to the other.

The task of so moving is neither mechanical nor easy That it sometimes takes a very long time to appreciate that a situation implies a problem is something that most parents experience through much of their child rearing. That problems can be neutralized (or de-posed as the title of this section playfully suggests) is something that may be equally difficult to appreciate Those of us who realize that we have been asking the wrong questions realize implicitly the need to move from a problem to a situation before re-posing the problem.

Consider the example of a "female response" to the Heinz dilemma which asserts with indignation that the problem is not one of stealing or not stealing the drug, but rather one of figuring out how we even evolved as a society such that such choices would have to be made (and one of figuring out how to reconstruct society) Here is a clear case of first neutralizing a problem before re-posing it It was necessary to delete the question (Should Heinz steal the drug?) before moving towards a re-posing of the problem.

It is not only that there is value in having students actually move in both directions—from situations to posing and from posing to de-posing—but it is also worth designing a curriculum which exhibits the difficulties people had in making such moves on their own in the history of discipline. We have the potential to learn a great deal about the relationship of a discipline to the culture from which it emerges as we study those problems that could not be perceived as situations.

An obvious example in the history of mathematics is that of efforts over several centuries to try to prove the parallel postulate Consider the following formulation of the question:

> How can you prove the parallel postulate from the other postulates of Euclidean geometry?

We know now that a great deal of the history of mathematics was written as nineteenth century mathematicians began to appreciate that the difficulty in solving the problem was that a wrong question was being posed In some implicit sense, Lobachevsky and his colleagues at the time had in fact to "neutralize" the problem enough first to get clearly at the *situation* from which it derived (the postulates of Euclidean geometry) and then to reformulate the question so as to delete the deceptively innocent word "How" in the posing of the problem.

The need to re-pose a problem by first neutralizing it is not only revealed through frustrated efforts at solving problems, but is an aesthetic issue as well, and an issue that is worth incorporating explicitly in curriculum within which the Gilliganish concept of context boundedness is taken seriously Consider the case of efforts to prove the four color conjecture—roughly that for any conventional map, four is a sufficient number of colors to establish and to appropriately demarcate boundaries. Until recently the problem was "merely" to prove or disprove that conjecture. Only after a computer proof was produced which featured a very large number of special cases did mathematicians begin to realize that they had not adequately posed the problem Feeling that a computer proof was blind to underlying structure and in fact illuminated very little of "the mathematical essence" of the problem, many mathematicians realized the need to state the problem in such a way that "ugly" proofs would not count as solutions.

Such re-posing of the four color problem reveals something not only about the present attitude of many mathematicians with regard to the computer, but just as importantly, it unearths some fundamental epistemological issues—issues that more clearly locate knowledge within an aesthetic realm.

From a pedagogical point of view, it is particularly enlightening to engage students in a discussion of the relationship of a situation to a problem. I have a modest example. Several years ago my son, Jordan, came to me to tell me that he did not understand the "ambiguous case" in trigonometry, i.e., those circumstances under which a triangle is determined by an angle, another angle and a side not included between the angles.

I began my discussion with him by asking him to recall how in geometry, he had investigated those conditions under which a triangle was determined. Jordan looked very puzzled and told me that I was mistaken; they had never investigated the determination of a triangle Instead they had proven things about two triangles being congruent if A.S.A. = A.S.A and so forth.

What was taking place here is very interesting from the point of view of relating a problem to a situation Jordan had in fact viewed an entire unit of work more as a *situation,* while I had viewed it as a *problem*. That is, though he had an arsenal of congruence theorems at his disposal to respond to any request to prove two triangles congruent, he did not see this ammunition as providing answers to what I saw to be the fundamental problem of discovering those conditions under which a triangle is determined As I reviewed his text, I understood why he saw a situation in what I saw to be a problem The book had in fact never distinguished between an underlying problem (determining a triangle) and a collection of exercises to give one experience in handling a problem that had been solved by the famous congruence theorem. In fact, the practice exercises had become the fundamental concept—a phenomenon I am beginning to believe is more widespread than I had thought, and a consequence most likely of the essentially plagiaristic spirit that governs textbook writing.

The interesting irony in this case is that the difference between my perception and Jordan's regarding what those congruence theorems were all about, was not revealed in Jordan's performance in geometry at all One can frequently accurately answer questions and even solve difficult problems without seeing the context within which those problems are embedded

Thus, it would seem to be a very wise pedagogical ploy to move not only from situation to problem and back for topics that are relatively small (e.g., de-pose a theorem such as "The base angles of an isosceles triangle are congruent"), but to do so for entire units as well. Teachers as well as students would find it enlightening to discover the areas of agreement and divergence of opinion regarding the problem/situation status of a unit or perhaps even of a course.

In closing this section, I would like to comment on an interesting potential difficulty relating situation to problem that Peter Hilton alluded to in his talk. He mentioned that proper selection of problems is critical in designing curriculum for one does not want to give problems to students that they are not prepared to handle. His comment appears on the surface to be a threat to the activity of posing and de-posing problems. That is, what happens if in the creation of a problem from a situation, a student defines a problem that we know is beyond his/her ability to handle?

There are some interesting assumptions embedded in the above question. First of all, it is not necessarily the case that students need to try to solve problems they pose. The activity of posing itself in the absence of efforts to solve may be illuminating both to students and teachers. In a sense we find out as much of value about ourselves by attending to the kinds of questions we ask as we do by the solutions we attempt.

Secondly, if we think of an entire class as a resource, for the kinds of activities suggested in this section, it is not necessarily the case that the same person who poses a problem need be obligated to try to solve it. In fact we may discover the potential for unexpected collaboration among those who pose and those who attempt to solve. We do not know very much at all about the relationship of the talent of posing and solving, but it perhaps is worth taking a clue from the work of Getzels and Jackson [1961] in which they find reason to conclude that beyond a certain point, intelligence and creativity may not be as closely related as one might suppose.

But there is another consideration that cuts deeper than those we have mentioned so far. That is, what do we imply students are incapable of doing when we say they are prematurely challenged? It appears that they may be incapable of *solving* problems that either we (as teachers) or they pose. But such an expectation may be a short-sighted one from an educational point of view however. Along with our newly discovered appreciation for the role of approximation and estimation, ought to come an appreciation for partial solutions as a respectable activity. We need not necessarily expect a complete solution for every problem investigated. In addition, I am not clear on what it is that is lost if students attempt to solve a problem and cannot even come up with partial solutions. Suppose they cannot even identify or isolate lemmas that might help them along the way. I can imagine a great deal of valuable personal and intellectual insight that might emerge through a discussion of what may account for inability of student at a particular point in time to make headway in solving particular problems. A teacher who keeps an ear to the ground might possibly even learn something of the students' conception of the subject matter, proof, mathematics and the relationship of mathematics to culture by listening carefully to what counts as a reason for failure to make headway.

VII The act of posing: logic and pedagogy

In relating problem posing to the creation of situations, we have, beneath the surface bumped up against the relationship of problem posing to problem solving. After all, it was due to an inability to solve the parallel postulate problem that a situation was revealed which was in need of reformulation. Problem generation and problem solving are intimately connected, however, even when things do not go awry. Below we discuss their intimate logical connection. In the two subsections that follow the one below we shall look more closely at pedagogical strategies for engaging in problem posing—one mild and the other radical. Much of what I will be analyzing in this section has appeared in disparate sources, and I view the task here as one primarily of consolidating that material. For that reason this section will be briefer than the others (thank God!) and the reader's attention will be drawn to relevant references for expansion of the points alluded to.

Logical connections with solving: being gracious and accepting

Consider the following two problems:

> (1) A fly and train are 15 km apart. The train travels towards the fly at a rate of 3 km/hr. The fly travels towards the train at a rate of 7 km/hr. After hitting the train, it heads back to its starting point. After hitting the starting point, it once more heads back toward the train until they meet. The process continues. What is the total distance this fly travels?
>
> (2) Given two equilateral triangles, find the side of a third one whose area is equal to that of the sum of the other two.

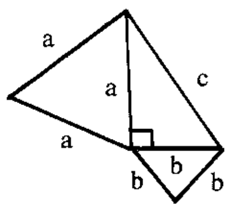
The first problem reveals in a dramatic way something that is true but less obvious in the solution of any problem. If you have not seen this problem before, let it sit for a while, or perhaps share it with a fifth grader. If the wind is blowing properly, you will come upon an insight that will most likely jar and inspire you. Without giving the ballgame away completely, let me suggest that an insightful and non-technical solution depends upon your asking a question that has not been asked in the problem at all. Though there are many different ways of asking the question as well as many questions to ask, something like the following will most likely be revealing:

> What do I notice if I focus not on the fly as requested, but on the train instead?

What is needed in the solution of this problem is some effort at posing a new problem within the context of accepting and trying to solve a given problem. Whether or not such a problem posing is *always* needed in the solution of a problem is an interesting and debatable question. I believe that such problem generation is always needed, but I also believe that the analysis of the assertion very much hinges on how it is that one defines a problem in the first place. [See Brown, 1981a; Brown and Walter, 1983 for additional discussion of this point]

The second problem reveals another interesting intimate connection between problem solving and problem generating. The solution depends (an illustration of what we have said above) upon how it is that the problem itself is redefined. If, however, you assume that sides and their lengths can be distinguished from each other (something that is *not* necessary in the solution of the problem), then if the lengths of the sides of the first two triangles are a and b respectively, we can prove without too much fanfare that the length of the third side c is equal to $\sqrt{(a^2 \times b^2)}$

Now in one sense we have solved the original problem. In another sense, however, we have only begun to solve it. Most people who come upon the solution, $c = \sqrt{(a^2 \times b^2)}$ are taken aback. The point is that it smells as if this is an interesting and unexpected connection (as a matter of fact one which now enables one to solve the problem without associating the sides with their lengths). The fact that the relationship is a Pythagorean one, indicates that we can find the third side as suggested below:



Most mathematicians who have not seen this problem before find themselves headed in an almost compulsive search for what is happening. They are driven by some variation of the question:

> I know areas are additive for the squares on the sides of the right triangle, but *why* are they additive for equilateral triangles as well?

What this example illustrates very nicely is that a proof or a solution in itself does not always reveal *why* things operate as they do Something more is needed, and in this case that something more begins with a question.

Though it is surely the case that the alleged solution of any problem always has further implications that one may assert as a problem or a question, one is not necessarily driven to do so in all problems with the same kind of fervor as in this case. [See Walter and Brown, 1977 and Brown and Walter, 1983 for an elaboration of this discussion]

There are pedagogical implications that flow from these relationships between posing and solving problems. Students are not always aware of the questions they may have implicitly asked themselves in coming up with the solution to a problem, and there might be value in encouraging them to explicitly see what they have done At the other end of the spectrum, students may not at all be aware of additional questions they "need to" or might ask after they have supposedly solved a problem

On strategies for posing an accepting mode

It is one thing to suggest that problem posing is worthwhile, or even necessary; it is another to be able to do it. We shall in this subsection suggest several strategies for posing problems, some of which are well discussed in the literature, and some of which represent new directions In this subsection and the next, we shall look at the activity of problem generation in a mode that is somewhat isolated from that of solving a problem that has already been stated. In so doing, we return to situations as a starting point. Much of what we do here might be appropriate to apply to the activity of solving an already stated problem as well. [See Brown and Walter, 1983 for an elaboration of these two subsections]

What are the "things" that situations are made of? Among possible candidates are the following:

1. *Concrete objects* like Cuisenaire rods and the Tower of Hanoi
2. *Abstract "things"* like
   (a) isosceles triangles or
   (b) nine Supreme Court justices each shaking hands with each other
3 *Data* like
   (a) primitive Pythagorean triplets generated by the relationships $x^2 + y^2 = z^2$ or
   (b) 5, 12, 19, 26, 33 . .
4. *Theorems or postulates* like the Fundamental Theorem of Arithmetic (every number can be expressed uniquely as a product of primes)

There are surely more kinds of "things" that one might use as a starting situation, but the above should serve the purpose of enabling us to see the directions we might look towards in generating questions

(i) Estimation/approximation

Here is a category with which we are all familiar, though we tend not to make as much use of it in practice as we might. Given phenomenon 2(b) for example, most people with a little knowledge will ask: How many handshakes are there? Of course it is just as illuminating (for some purposes) a question to ask: *About* how many handshakes are there?

(ii) Internal and external views of a thing

Given situation 2(a), most people will ask the rather familiar question: What can you say about the base angles? Some people might extend the base and ask about the external angles. Compare those kinds of questions with one like:

> How many isosceles triangles can you join to form the hub for the bicycle wheel?

How does the above question differ from the other isosceles triangle questions? It is worth pointing out that while the first set focuses on the internal working of the phenomenon, the one dealing with the hub takes the isosceles triangle in its entirety and relates it to something else. Much of our standard curriculum is focused on an internal view of objects and relatively little takes as its startng point the object as a whole.

(iii) The particular and the specific

Here is a theme that is particularly salient in terms of a Gilligan perspective. Take a look at 3 above and pose some problems.

Our enchantment with abstraction and generalizability frequently blinds us from seeing the uniqueness of what is before us. Most people shown 3(a) and (b) will pose a problem that attempts to reveal some covering law that will generate all the terms. A careful look at data, however, frequently suggests that there is more to see that might be equally as appealing. Consider the following for example with regard to 3(a):

> Each triplet has at least one member divisible by 3, by 4, and by 5 Will that hold in general?

The above is clearly *not* a question that would arise if our focus were upon the more abstract Pythagorean relationship

Take another look at 3(b) What questions arise from a careful look at the data beyond a search for some general algebraic generating formula?

(iv) On pseudo-history

Many teachers wish they knew more about the history of mathematics so that they might be better able to motivate the subject What is not well appreciated, however, is that a great deal of intellectually stimulating thought can flow from an effort on the part of students as well as teachers to engage in what I call pseudo-history [Brown, 1978 as well as Brown and Walter, 1983]. As an example, consider the following kind of question conceivably generated by 4.

> What *might* have been responsible for getting people to look at products of primes.

We can, for example, imagine a mathematics community that focused originally on expressing any given number as the *sum* of other numbers What might have moved them to look at *products* instead?

These are surely not the only categories for generating problems while at the same time maintaining an accepting view towards the beginning situation They do, however, represent a start, and with the exception of (i) tend not to be given much curriculum consideration. It would be a valuable contribution to expand both the list of "things" upon which one might generate questions as well as the categories one might towards in the generation of questions

Posing as an adolescent

In this subsection we further expand the concept of problem generation by selecting a situation in a mode that is more reminiscent of adolescent rebellion than is the previous subsection. Though perhaps the most intriguing, this is the aspect of problem posing that Marion Walter and I have written more about than any other, and I therefore will be brief and suggest that you refer to relevant pieces cited if you wish further elaboration.

The concept of challenge, threat or adolescent rebellion is well captured by Hofstadter [1982] when he comments:

> George Bernard Shaw once wrote (in Back to Methuselah): "You see things; and you say 'Why?' But I dream things that never were; and I say 'Why not?' " When I first heard this euphuism, it made a lasting impression on me. To "dream things that never were"—this is not just a poetic phrase but a truth about human nature. Even the dullest of us is endowed with this strange ability to construct counterfactual worlds and to dream. Why do we have it? What sense does it make? How can one dream or even "see" what is visibly not there? . . . Making variations on a theme is really the crux of creativity On the face of it the thesis is crazy. How can it possibly be true? Aren't variations simply derivative notions, never truly original creation? [p. 20] . . . Careful analysis leads one to see that what we choose to call a new theme is itself always some kind of variation, on a deep level of earlier themes [p. 29]

One can start with a definition, a theorem, a concrete material, data, or any other phenomenon and instead of *accepting* it as the given to be explored, one can challenge it and in the act create a new "it."

> Consider for example the definition of a prime number: A natural number is prime if it has exactly two different divisors.

Now the "natural" inclination of the standard curriculum is to use that bit of information to prove or show all kinds of things. An adolescent rebellion on the other hand might generate a host of questions like:

> What's so special about numbers that have exactly *two* different divisors? What kinds of numbers have exactly three divisors?
>
> Why do we focus on *divisor?* Can we find numbers that have exactly two different elements to form a sum?
>
> Why are we focusing on *different* divisors? Can numbers have the *same* divisor twice?
>
> Why do we focus on *natural* numbers? Suppose we look instead at fractions or the set of odd integers.

I shall not continue with the list of such questions that can be generated to challenge rather than accept the concept of prime number [See Brown, 1978, 1981 for a thorough development especially of the last question] Let me merely indicate that such activity has a built-in kind of irony, for it is in the act of "rebellion" that one comes to better understand the "thing" against which one rebels. In that sense challenging "the given" as a strategy for problem generating has the potential to be viewed as a less radical departure from standard curriculum than one might otherwise believe.

Marion Walter and I have taken the insight of challenging the given and created a scheme which we call "what-if-not." A number of people both within the CMESG group and outside of it, have derived some fascinating and imaginative concepts by employing the scheme Though it is possible to approach that scheme in an overly mechanistic manner, it is also something that can be done with taste.

Suppose one wishes to do a "what-if-not" on the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

For the *first* stage of the scheme, one lists the attributes of "the thing," without worrying about such matters as completeness, repetition, elegance of statement, independence of statements and so forth. Thus we might list among the attributes:

1. The sequence begins with the same first number
2. The first two numbers are 1.
3. If we do *something* to any two successive terms, we get the next number in the sequence
4. The something we do is add

At the second stage, we do a "what-if-not" (hence the name of the scheme) on one of the attributes. For example, suppose we do a "what-if-not" on number 2 above; if it is not the case that the first two numbers are 1, we ask what they might be? Obviously we could select many alternatives to 1 and 1 as the starting numbers. Suppose we chose 3 and 7

At the next stage, we ask some new set of questions about the modified phenomenon. Suppose we begin by asking what the new sequence would look like. To continue the process, we finally engage in the kind of activity which most people incorrectly assume is the essence of mathematics—namely we analyze or try to answer the question. Thus, if we maintain the essential definition of the original sequence (something we need not necessarily feel obligated to do), we would get:

3, 7, 10, 17, 27, 44, 71, . . .

Moving back to the stage of asking some new set of questions, we might ask:

- Is there an explicit formula to generate the *n*th term of the sequence?
- How do properties of this sequence compare with those of the original one?

An analysis of these questions reveals some *very* fascinating jewels. People who are familiar with properties of the original Fibonacci sequence, in analyzing the second question above, most likely would look (among other things) at ratios of succeeding terms. Choosing smaller to larger adjacent terms, we would get:

.42, .70, .588, .708, .614, .62

Something smells (as in the equilateral triangle example in the previous subsection) peculiar. We are arriving at ratios that appear to be very close to the "golden ratio" (approximately .618)—something we expect from the original Fibonacci sequence. Why is that happening?

In analyzing the question above, one is thrown back towards an analysis of the original phenomenon—as we indicated above.

We have barely begun to see the wealth of surprising results in making use of the "what-if-not" strategy of the Fibonacci sequence [See Brown, 1976 and Brown and Walter, 1983 for a more detailed discussion]. In this brief sketch, however, we implied the value both of carefully employing the various stages of the "what-if-not" strategy and of interrelating them as well.

In closing, it is worth pointing out that despite efforts to mechanize the stages, the process tends to elude a computerized mentality, for it is frequently the case that in the absence of an essentially human activity one may never even "see" some of the attributes to vary in the first place. Elsewhere [Brown, 1971, 1974, 1975, 1981] I have shown how it is that use of poetic devices such as metaphor and imagery, and such human qualities as finding surprise and flipping figure and ground frequently account for our ability to see what it is that is supposedly staring us in the face all along.

References

Brown, Stephen I. "Rationality, irrationality and surprise." *Mathematics Teaching,* No. 55, Summer 1971, pp. 13-19

Brown, Stephen I. "Mathematics and humanistic themes: sum considerations," *Educational Theory* Vol. 23, No. 3, 1973, pp. 191-214

Brown, Stephen I. "Musing on multiplication." *Mathematics Teaching* No. 61, December 1974, pp. 26-30

Brown, Stephen I. "A new multiplication algorithm: on the complexity of simplicity," *The Arithmetic Teacher* Vol. 22 No. 7 November 1975 pp. 546-554

Brown, Stephen I. "From the Golden Rectangle and Fibonacci to pedagogy and problem posing," *Mathematics Teacher.* Vol. 69. No. 3, March 1976, pp. 180-186

Brown, Stephen I. *Some "prime" comparisons.* Reston, Virginia: National Council of Teachers of Mathematics, 1978

Brown, Stephen I. "Some limitations of the structure movement in mathematics education: the meaning of 'why'" *Mathematics Gazette of Ontario.* Vol. 17, No. 3, March 1979, pp. 35-40

Brown, Stephen I. "Ye shall be known by your generations." *For The Learning of Mathematics.* No. 3, March 1981, pp. 27-36

Brown, Stephen I. "On humanistic alternatives in the practice of teacher education," *Journal of Research and Development in Education* Vol. 15, No. 4, Summer, 1982, pp. 1-12

Brown, Stephen I. "Distributing isomorphic imagery," *New York State Math Teachers Journal,* Vol. 32, No. 1, Winter 1982a, pp. 21-30

Brown, Stephen I. "Problem posing: the problem generation gap." *Math Lab Matrix,* No. 16, Fall 1981a, pp. 1-5

Brown, Stephen I., and Walter, Marion I. *The art of problem posing.* The Franklin Institute Press, 1983

Dewey, John. *Reconstruction in philosophy.* Henry Holt and Co., N.Y. 1920

Dewey, John. *How we think,* Heath, N.Y., 1933

Getzels, Jacob W., and Jackson, Philip W. *Creativity and intelligence: explorations with gifted students.* John Wiley & Sons, N.Y., 1961

Gilligan, Carol. *In a different voice.* Harvard University Press, Cambridge, 1982

Higginson, William C. *Toward mathesis: a paradigm for the development of humanistic mathematics curricula.* Unpublished doctoral dissertation, University of Alberta, 1973

Hofstadter, Douglas. "Metamagical themas: variations on a theme as the essence of imagination," *Scientific American,* October, 1982, pp. 20-29

Kohlberg, Lawrence. "Moral stages and motivation: the cognitive developmental approach," in Lickona, Thomas (editor), *Moral development and behavior: theory, research and social issues.* Holt, Rinehart and Winston, 1976, pp. 31-53

Lickona, Thomas (editor). *Moral development and behavior: theory, research and social issues,* Holt, Rinehart and Winston, 1976

Lipman, Matthew, Sharp, Ann S., Oscanyan, Frederick. *Philosophy in the classroom,* Institute for the Advancement of Philosophy for Children, New Jersey, 1977

Lyons, Nona P. "Two perspectives: on self, relationships and morality," *Harvard Educational Review,* Vol. 53, No. 2, May 1983, pp. 125-145

McGinty, Robert L., and Meyerson, Lawrence, N. "Problem solving: look beyond the right answer," *Mathematics Teacher.* Vol. 73, No. 7, October 1980, pp. 501-503

Oliver, Donald, and Newman, Fred. *Clarifying public controversy: an approach to teaching social studies,* Little, Brown, 1970

Snow, C. P. *The two cultures: and a second look.* The New American Library, New York, 1959

Taba, Hilda. "The problems in developing critical thinking." *Progressive Education,* 1950, pp. 45-48

Walter, Marion I., and Brown, Stephen I. "Problem posing and problem solving: an illustration of their interdependence." *Mathematics Teacher,* Vol. 70, No. 1, Fall 1977, pp. 4-13

Walter, Marion I., and Brown, Stephen I. "What if not?" *Mathematics Teaching,* No. 46, Spring 1969, pp. 38-45

Wheeler, David. "Mathematization matters," *For the Learning of Mathematics,* Vol. 3, No. 1, July 1982, pp. 45-47