USING WARRANTED IMPLICATIONS TO UNDERSTAND AND VALIDATE PROOFS

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Implication is one of the most basic structures for understanding mathematical truth (Rodd, 2000) and accurately interpreting implication is essential for understanding and producing proofs (Hoyles and Küchemann, 2002). However, implication is also a topic that causes students serious difficulties (*e.g.*, Hoyles and Küchemann, 2002; Deloustal-Jorrand, 2002; Durrand-Guerrier, 2003).

One cause of students' troubles is that the meaning of implication is not straightforward; psychologists and mathematics educators have observed that different conceptions of implication are used in different contexts (*e.g.*, Deloustal-Jorrand, 2002; Durrand-Guerrier, 2003). For instance, Mitchell (1962) defines *material implication* as part of propositional logic; "if p, then q" is true whenever the consequent q is true or the antecedent p is false. Mitchell also defines a *hypothetical proposition* to only assert the consequent q is true when the antecedent p is realized; cases where p is false are irrelevant and ignored. It has been argued that while the material conception of implication is the one most commonly used in formal mathematics, the latter conception actually plays a more important role in school mathematics (Holyes and Küchemann, 2002).

Each of the schemes described above is useful for determining whether or not an implication is *true*. The purpose of this article is to argue that, when reading a proof, one needs to determine not only whether each implication it uses is true, but also whether these implications are *warranted*. Specifically, we will:

1. define and illustrate what we mean by a warranted implication
2. argue that this theoretical construct is useful for conceptualizing the way in which individuals understand and validate mathematical proofs
3. maintain that the skill of inferring and evaluating warrants when validating proofs should be taught to students.

Mathematicians' behavior while judging whether a given proof is valid

Our investigations into the nature of implication began with the following observation. Consider the statement:

If 7 is prime, then 1007 is prime.

Direct computation can be used to show that 1007 is prime. Hence, this statement is true using a material conception of implication. Now consider the following argument that purports to prove that 1007 is prime.

Proof. 7 is prime.

If 7 is prime, then 1007 is prime.

Therefore, 1007 is prime.

We presented this argument to two mathematicians who regularly taught transition-to-proof courses and observed them as they determined whether it constitutes a valid proof. Both rejected the proof as invalid. Relevant transcripts of their responses are provided below [1]:

P1: The second statement seems to be using some general principle. Which it hasn't stated what the general principle is. The reader is left to guess what the general principle is. So I guess the point is that the statement, "if 7 is prime then 1007 is prime" is logically valid. I don't like if-then statements of this form. The point is that, implicit in that, when we use that in a proof, there's really some general principle we have in mind. So for instance like, "if x is prime, then $1000 + x$ is prime". That general principle is false. So the question would be, what is the general principle that you are using to deduce that? And I can't think of one.

P2: [reading the proof] "7 is prime", I believe that line. "If 7 is prime, then 1007 is prime" – I have no reason to believe that line yet. In the absence of implication, I certainly can't make the inference. Why would I believe that the sum of 1000 and 7 should be prime if 7 is prime? I've no idea. [...]

Okay, I'm going to believe that the "If 7 is prime, then 1007 is prime" is true. If I believe that it is true, then I would believe the conclusion. [The implication is] valid, but not informative.

We also presented the same mathematicians with the following argument purporting to prove that $(\sqrt{n}) \to \infty$ as $n \to \infty$.

Proof. We know that if a, b, and m are positive numbers and $a < b$, then $a^m < b^m$.

So $a < b \Rightarrow \sqrt{a} < \sqrt{b}$, for all positive a and b.

For all natural numbers n, $n > 0$ and $n + 1 > n$, so $\sqrt{n} < \sqrt{n+1}$.

So $(\sqrt{n}) \to \infty$ as $n \to \infty$ as required.

Excerpts from the mathematicians' comments while reading this proof follow:

P1: I see. So I mean there's a problem that a monotone sequence does not necessarily tend to infinity. That's certainly missing [...] I think it's invalid. [...]

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I mean the statement [the last line of the argument] is true [...] Again there is a general principle, which you know you always [are using] in deducing the next statement. You are using a general principle that is either a logical principle or some theorem that you are supposed to know [...] Now this [the last line of the argument] appears to be invoking a false general principle.

P2: [reading the argument] "So square root of n approaches infinity as n approaches infinity as required". Not shown. There are lots of increasing sequences that do not approach infinity. [...] All that I believe is that it's shown that the sequence square root n is monotone increasing, and in fact it's strictly monotone increasing. But even that's not enough to have it approach plus infinity.

There are important commonalities in the mathematicians' behaviors that we would like to emphasize. First, when the mathematicians came to the assertion "If 7 is prime, then 1007 is prime", they did not appear to be initially concerned with whether this implication was true. Rather, they focused their attention on what general principle the author of the proof was using to deduce that 1007 is prime from the fact that 7 is prime. Since no such principle was explicitly given, both mathematicians attempted to infer such a principle (*e.g.*, if x is prime, then $1000 + x$ is prime). As it seemed that no valid principle was available, both rejected the proof as invalid, even though they believed the implication was logically true.

Likewise, both mathematicians rejected the second proof because they inferred that the author implicitly used the fact, "If a sequence is strictly increasing, it diverges to infinity", which is invalid.

In short, when reading an implication in the context of a proof, the mathematicians were not only concerned with whether the implication was true, but also whether the implication was warranted – *i.e.*, whether there was a legitimate mathematical reason for asserting that the conclusion of the implication was a consequence of its antecedent. Since a warrant was not explicitly provided in either of these proofs, the mathematicians attempted to infer an appropriate one. In each case, the mathematicians did not find a satisfactory warrant, and this led them to reject each proof as invalid.

Framework for comparing interpretations of implication

In this section, we will discuss the differences between evaluating the truth of an implication and determining whether or not an implication is warranted. To do so, we will use Toulmin's model of argumentation (Toulmin, 1969). Toulmin's model specifies what features one should attend to in evaluating the acceptability of a scientific argument.
Krummheuer (1995) introduced this model to researchers in mathematics education as a way to understand and evaluate informal mathematical arguments and formal proofs. We argue that there is a qualitative difference between what one should attend to in evaluating whether an implication is true and in determining whether an implication is valid, and we will use Toulmin's model of argumentation to make these differences clear.

Toulmin's model of argumentation

According to Toulmin (1969), an argumentation consists of at least three essential parts called the core of the argument: the *data*, the *conclusion*, and the *warrant*. When one presents an argument, one is trying to convince an audience of a specific assertion. In Toulmin's framework (see Figure 1), this assertion is referred to as the *conclusion*. To support the conclusion, the presenter typically puts forth evidence or *data*. The presenter's explanation for why the data necessitate the conclusion is referred to as the *warrant*. At this stage, the audience can accept the data but reject the explanation that the data establishes the conclusion – in other words, the authority of the warrant can be challenged. If this occurs, the presenter is required to present additional *backing* to justify why the warrant, and therefore the core of the argument, is valid.

Material implication and truth

Using a material conception of implication, a statement of the form, "if p, then q" is equivalent to stating "not p or q". Hence, when one is asked to verify such a proposition, one needs to demonstrate either that q is true or that p is false. In Toulmin's scheme, the statement "if p, then q" itself is the conclusion. The data to support this conclusion can either be "p is false" or "q is true". The warrant for this statement is the formal logical equivalence between the statements "if p, then q" and "not p or q". (Obviously the situation is more complex for quantified conditional statements, such as, "For every integer n, if n is odd, then n^2 is odd". An analysis of

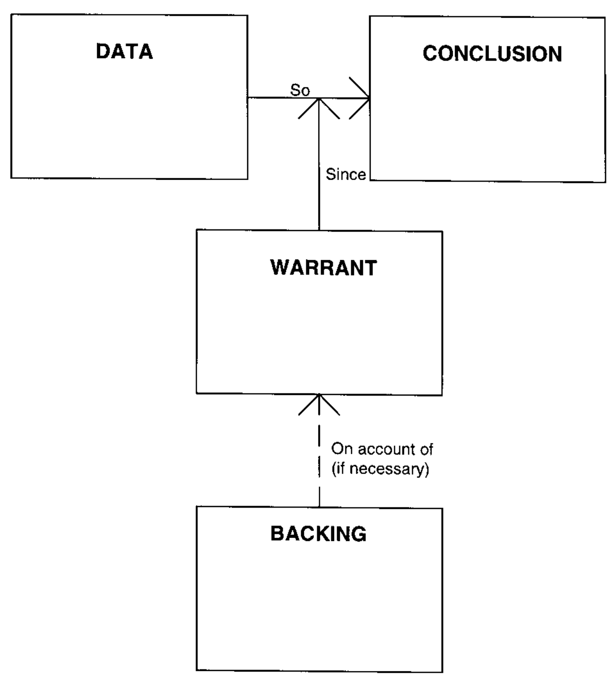

Figure 1: Toulmin's argumentation model.

what is needed to satisfy this type of statement is given in Durrand-Guerrier (2003)).

Using this material conception, the example given in the previous section, "If 7 is prime, then 1007 is prime" is itself the conclusion. The data is "1007 is prime". The warrant is "If q is true, then 'If p, then q' is true" (see Figure 2 for a diagram illustrating this argument).

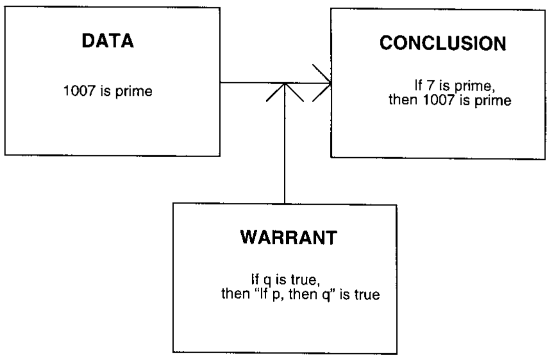

Figure 2: An argumentation of the material implication, "If 7 is prime, then 1007 is prime".

There are important limitations in evaluating implications in this manner. In practice, and especially in proof writing, implications are often presented to convince the audience that the conclusion of the implication follows as a consequence of its antecedent. However, when determining the truth-value of a material implication, one does not consider any relationship between its antecedent and conclusion. Indeed, a statement of the form, "If p, then 1007 is prime", will be true for any statement p, regardless of the truth value of p and regardless of any direct or indirect relationship between p and the fact that 1007 is prime.

Warranted implication and validity

Recall that the mathematicians rejected our first proof because of the assertion, "If 7 is prime, then 1007 is prime". The mathematicians did not argue that this implication was false - on the contrary, they averred that it was logically true. Rather, the fault that they found with this implication was that the general principle used to deduce its conclusion from its antecedent was invalid.

We use Toulmin's scheme to frame this in the following way. When one evaluates whether the implication "if p, then q" is warranted, p is seen as the data and q as the conclusion. If no justification is explicitly given for drawing this conclusion on the basis of this data, the reader is left to infer a warrant. In determining whether "if p, then q" is warranted, the reader must not only evaluate the truth of p and q, but also judge the soundness of this possibly inferred warrant. In the case of "if 7 is prime, then 1007 is prime", the data is "7 is prime" and the conclusion is "1007 is prime". One of the mathematicians that we spoke with (quite sensibly) inferred that the prover's warrant in this case was, "If x is prime, then $1000 + x$ is prime" (see Figure 3 for a diagram illustrating this argumentation). Of course, this warrant is invalid (*e.g.*, 5 is prime, but 1005 is not), leading the mathematicians to reject the proof that depended upon this statement.

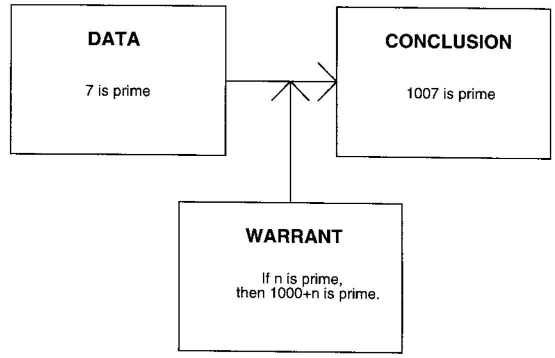

Figure 3: An argumentation of the causal implication, "If 7 is prime, then 1007 is prime".

Using warranted implications to discuss understanding and validation of proofs

To fully understand a mathematical theorem, Rodd (2000) argues that it is not sufficient simply to believe the theorem is true. One must also have a legitimate mathematical justification for this belief. In the mathematical community and advanced mathematics courses, such justifications usually take the form of formal proofs. To be fully convinced that a proof establishes that a theorem is true, an individual must have the ability to distinguish between arguments that prove the statement in question and arguments that do not. Selden and Selden (2003) define *validation* as the act of judging whether an argument constitutes a legitimate proof of a statement (*i.e.*, whether a proof is valid). These authors stress that the ability to validate proofs is critically important for both mathematicians and students of mathematics. If an individual cannot reliably determine if an argument proves a theorem, some other statement, or nothing at all, that argument cannot legitimately convince the individual that the theorem is true (*cf.* , Selden and Selden, 1995). In this section, we argue that to determine whether or not the proof is valid, it is necessary to consider whether the implications used in the proof are warranted.

The inadequacy of material implication for proof validation

To illustrate the inadequacy of using a material conception of implication in validating proofs, consider the following two argumentations:

> *Argumentation 1*: Since x^3 and x^2 are continuous functions, $x^3 + x^2$ is a continuous function.
>
> *Argumentation 2*: Since x^3 and x^2 are continuous functions, $x^3/x^2 = x$ is a continuous function.

We claim that the first argumentation would be permissible for a student in a real analysis course to use in a proof (assuming that it has been established that the functions $f(x) = x^3$ and $g(x) = x^2$ are continuous), while the second argumentation would not. However, when using a material conception of implication, it is difficult to see why we should favor one argumentation over the other. In our view, each hinges on the acceptance of the following respective implications, which have nearly identical formal structure:

Implication 1: If x^3 and x^2 are continuous functions, then $x^3 + x^2$ is a continuous function.

Implication 2: If x^3 and x^2 are continuous functions, then $x^3/x^2 = x$ is a continuous function.

In both implications, the antecedent and consequent are true. Further, it is most likely that neither implication has been explicitly proved by the professor in their lectures or by the students in their prior work. From a strictly logical perspective, differences in their acceptability are difficult to distinguish. [2]

Our framework highlights the differences between these implications. A sensible inferred warrant for the first implication would be that, "the sum of two continuous functions is continuous". As this warrant is an established theorem in real analysis, the first implication is a warranted implication.

The warrant for the second implication appears to be, "If $f(x)$ and $g(x)$ are continuous, then $(f/g)(x)$ is a continuous function". This warrant is not true (*e.g.*, $f(x) = x$ and $g(x) = x^2$ are both continuous, but $(f/g)(x) = 1/x$ is not). Hence, the second implication is not warranted, and it is this that makes the argumentation unacceptable in a proof.

A framework for validating proofs

Many proofs in mathematics journals and textbooks are presented as a series of assertions. Often it is the case that some of these assertions are not explicitly justified [3]. In order to establish that a particular assertion in a proof is valid, we argue that the reader should determine whether the implication, "If (a subset of the previous assertions in the proof), then (new assertion)", is warranted. If a warrant for this statement is not explicitly provided, the reader is left to infer it. Consider the following generic proof, where A, B, C, and D represent mathematical assertions.

Statement. If A, then D.

Proof. Assume A.

B

C

Therefore, D

We would argue that validating the third line of the proof would require the reader to determine if the implication, "If A and B, then C", was warranted. Note here that simply using a material conception of implication to see if this implication was true would not be sufficient. Material implications will *always* be true if the consequent of the implication is true. Hence, *any* string of correct statements would constitute a valid proof, even if some of these statements were non-trivial assertions that did not follow from previous work. Also note that a material conception of implication would not be useful for gaining understanding of why a statement was true by reading its proof. The truth of an implication does not depend on the relationship between its consequent and antecedent.

Of course, proofs are rarely as sparse as the generic proof presented above. Proofs regularly contain words such as "since", "thus", and "now". Although these words do not have a specific logical meaning, they nonetheless serve an important function by alerting the reader to data and warrants for assertions that might not otherwise be inferable. For instance, "thus" implies that the data for the forthcoming assertion appears in the previous few lines.

We conjecture one's line-by-line verification of a proof might proceed like this. Each line of the proof is interpreted as an argumentation whose conclusion is the statement being asserted. The reader of the proof identifies the data and the warrant used in this proof, inferring them if necessary. If the warrant for the argumentation is socially agreed upon by the mathematical community, this line is accepted as valid. If the warrant is false, this line and the entire proof are declared to be invalid. If the warrant of an argumentation is plausible, but not socially agreed upon by the mathematical community, backing for this warrant is required and the proof is said to have a "gap" in it.

To illustrate this process, we use the above as a prescription for analyzing a proof from an undergraduate textbook on real analysis. This proof is taken verbatim from Abbot (2001), but is separated into numbered lines to facilitate its analysis.

Statement: If a sequence is monotone and bounded, then it converges

Proof.

1. Let (a_n) be monotone and bounded.
2. To prove (a_n) converges using the definition of convergence, we are going to need a candidate for the limit.
3. Let's assume that the sequence is increasing (the decreasing case is handled similarly) and consider the set of points $\{a_n: n \in N\}$.
4. By assumption, this set is bounded,
5. so, we can let $s = \sup\{a_n: n \in N\}$
6. It seems reasonable to claim that $\lim(a_n) = s$ [a number-line diagram is included at this point]
7. To prove this, let $\varepsilon > 0$.
8. Because s is the least upper bound of $\{a_n: n \in N\}$, $s - \varepsilon$ is not an upper bound,
9. so, there exists a point in the sequence a_n such that $s - \varepsilon < a_n$.
10. Now, the fact that (a_n) is increasing implies that if $n \geq N$, then $a_n \leq a_n$.
11. Hence, $s - \varepsilon < a_n \leq a_n \leq s < s + \varepsilon$,
12. which implies $|a_n - s| < \varepsilon$ as desired.

Lines 1, 3 and 7 of this proof are assumptions, so their validity need not be judged [4].

Lines 2 and 6 are motivational and could be omitted without altering the substance of the proof. Our analysis therefore focuses on the remaining lines.

Line 4: In this line, the conclusion is that the set $\{a_n: n \in N\}$ is bounded. The words "by assumption" refer to line 1, and seem to indicate that this conclusion is identical to this assumption. In our terms, however, the data would be that the sequence

(a_n) is bounded and one must infer the warrant that if a sequence is bounded, the set containing its terms is bounded.

Line 5: In this line, the data is the fact that the set is bounded. The word "so" indicates that this data appeared in the immediately preceding phrase. The conclusion is that the set has a supremum and for a warrant one must infer that any subset of the reals that is bounded above has a supremum in the reals, *i.e.*, that the reals are complete.

Line 8: In this line, the data is the fact $s - \varepsilon$ is greater than 0 and that s is the supremum of the set. The fact that the latter is data is explicitly stated via the "because" clause. The conclusion is that $s - \varepsilon$ is not an upper bound. The warrant is that there can be no upper bound for the set that is less than the supremum - the use of the alternative terminology "least upper bound" serves to emphasize that this is the warrant being used. One may also note that a sub-argument could be offered as backing for this warrant. This would have data that ε is greater than 0, warrant that if $x > 0$ then $s - x < s$ and conclusion that $s - \varepsilon < s$.

Line 9: In this line, the data is that $s - \varepsilon$ is not an upper bound. Once again, the use of "so" indicates that the conclusion from the previous line forms the data for this one. The conclusion is that there exists a point in the sequence a_n such that $s - \varepsilon < a_n$. For the warrant, one must recall the definition of upper bound (or, more precisely, its negation) and the fact that the set is the set of terms of the sequence.

Line 10: In this line, the data is that the sequence is increasing. The use of "now" indicates that in this case this data does not come from the conclusion of the previous line, but rather is being invoked from elsewhere (in this case, the as-yet-unused assumption in line 1). The conclusion is that if $n \geq N$, then $a_N \leq a_n$. The warrant appears to be that for every increasing sequence and natural number N, if $n \geq N$, then $a_N \leq a_n$. Backing for this warrant would involve invoking the definition of increasing and using an inductive argument.

Line 11: The "hence" at the beginning of this line seems to indicate that the conclusion uses data from the immediately preceding lines, but in fact one also needs the fact that s is an upper bound and that $\varepsilon > 0$. In this case, the warrant needed is that inequalities may be chained together in this way, backing for which would rely on the order axioms of the real numbers.

Line 12: This final line takes line 11 as data. This is explicitly stated via the use of the phrase "which implies that". The conclusion is that $|a_n - s| < \varepsilon$ (with some conditions on <italics on>N<italics off>, although this is not explicitly re-stated), and the warrant is the definition of absolute value.

The above illustrates how one can consider and evaluate warrants to read and validate a proof as presented in the textbook for an undergraduate mathematics course. We would argue that if warrants were not considered when reading the proof, one's understanding of the proof would be limited. For instance, if one did not consider the warrant in line 5 (*i.e.*, that sets of bounded real numbers had suprema), one would not see how the proof hinged in a critical way upon the completeness of the real numbers. One might believe that this proof establishes that bounded, monotonic sequences over any ordered field converged, although such an assertion is not generally true (*e.g.*, rational-numbered sequences need not converge to a rational number).

Pedagogical implications

What are the implications of the above discussion for the teaching of proof-oriented mathematics? Teaching for these mathematics courses often consists of the professor presenting proofs to establish theorems (*e.g.*, Davis and Hersh, 1981; Weber, 2004). In this article, we have argued that for students to gain conviction and understanding from these proofs, they must consider the implicit warrants used to justify the assertions in the proof. However, it is not clear that students will naturally do this.

This conclusion leads us to argue that instruction in proof-oriented mathematics courses should call attention to the processes of inferring and evaluating warrants. However, we have observed that the issue of warrants is not discussed in proof-oriented mathematics courses, at least not in a systematic way. Textbooks on logical reasoning introduce the notion of *material* implication in a highly structured manner (*e.g.*, in terms of rules for inference and truth tables), but usually give no explicit discussion of the need to consider warrants when reading proofs. Similarly, a detailed observation of the instruction of an introductory real analysis course also revealed that logical rules for implication were stressed, but the issue of warrants was largely ignored (Weber, 2004).

It might be useful to consider why teachers give explicit attention to whether an implication is true, while placing less emphasis on the issue of whether an implication is warranted. Assessing the truth of an implication is a well-defined task. One only needs to check the truth of the antecedent and the consequent and to confirm that either the antecedent is false or the consequent is true. If a student presents an implication that is not true, the professor can clearly indicate why it is false. Determining whether an implication is warranted is not such a well-defined task. One cannot provide a canonical procedure for inferring a warrant. Likewise, one cannot provide a well-specified set of guidelines for determining whether a warrant is legitimate. In fact, determining whether a warrant would be considered acceptable by the mathematical community may inherently involve a degree of subjectivity. If a student presents a true but unwarranted implication in a proof, it may be very difficult to convince the student that there is a good reason for rejecting their proof as invalid. This apparent subjectivity in a supposedly rigorous subject may naturally make both professors and students uncomfortable. It is perhaps for this reason that warrants are not emphasized in proof-oriented undergraduate mathematics courses.

Given that considering whether implications are warranted is essential for understanding proofs in advanced mathematics courses, we would argue that warranted implications should be discussed in a systematic manner in transition-to-proof courses. We believe that our framework as outlined above can form a basis for this discussion by providing both a theoretical means of highlighting the differences between material and warranted implication and a practical structure for the process of inferring and evaluating warrants.

[Notes and references can be found on page 51 (ed.)]

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These notes and references follow on from page 38 of the article "Using warranted implications to understand and validate proofs" by Keith Weber and Lara Alcock that starts on page 34 (ed.)

Notes

[1] Utterances such as "umm", stutters, and repeated words were removed from the transcripts that are quoted to increase their readability. The text […] denotes that short segments of the transcript were deleted.

[2] Perhaps one could argue that the first implication is more acceptable than the second because its proof is more obvious. However, this would require the reader to infer the author's intentions for why this would be easier to prove. Such an action, to us, would closely correspond to inferring a warrant.

[3] Note that this is not due to carelessness on the part of the author. It is widely acknowledged that proofs would be impossibly long if each logical detail was included (*cf.* Davis and Hersh, 1981).

[4] The reader instead should be concerned with whether the assumptions used are part of a legitimate proof structure or framework (*cf.* Selden and Selden, 1995).

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