On Psychology, Historical Epistemology, and the Teaching of Mathematics: Towards a Socio-Cultural History of Mathematics*[1]

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Something is happening to the way we think about the way we think.
C. Geertz, *Local knowledge*

During these last years, the history of mathematics has been increasingly used for teaching purposes. One of the very well-known ways of doing so is to relate historical anecdotes to students. Another way is to view the history of mathematics as a huge arsenal of chronologically ordered problems to be "imported" into the classroom and to have students solve them. Although some students' motivation may be gained through both actions, the use of the history of mathematics remains at a superficial level.

The aforementioned educational use of the history of mathematics changes radically when one sees the history of mathematics as a kind of epistemological laboratory in which to explore the development of mathematical knowledge. Among other things, this requires us to take a certain theoretical point of view justifying the *link* between historical and modern conceptual developments. Unfortunately, this question is quite often completely evaded — a tacit assumption being that the past knowledge must have *something* to do with the modern one. When things become more explicit, a supposed parallelism between ontogenesis and philogenesis is often mentioned.

The goal of this paper is to contribute to a reflection on the possibilities and the limits of a non-naïve use of the history of mathematics for educational purposes. In section 1, we deal with two problems of a historiographical nature that any epistemological research of the history of mathematics must tackle. In section 2, we discuss a problem (which is specific to didactic historico-epistemological inquiries) related to the hypotheses that make it possible to confront past and modern conceptual developments. We examine some assumptions underlying Piaget and Garcia's psychological version of Haeckel's law of recapitulation and submit a critique based on a socio-cultural conception of knowledge. In section 3, we propose a critical analysis of one of the most influential historico-epistemological perspectives used by researchers in mathematics education up until now — that of epistemological obstacles. In section 4, we further develop an idea that has been used in the previous sections (and on which our critiques are based): that knowledge is deeply rooted in and shaped by its social and cultural context. We provide some examples taken from Chinese mathematics and from the Classical Greek and Alexandrian periods. In section 5, we discuss the possibilities that a socio-cultural history of mathematics offers for teaching.

§1 Some historiographical problems.

When researchers in mathematics education turn to the history of mathematics, they often realize that the books present the history as a sequence of events that cannot necessarily answer their epistemological questions. In fact, very often, the books unfold episodic narratives implicitly underlain by an apriorist epistemology of platonistic style. This leads us to see past mathematical achievements as clumsy efforts that always tended to the conceptual formulation that we find in our modern mathematics.

Let us give just one example. When Diophantus' *Arithmetica* is seen from this apriorist platonic epistemological perspective, this monumental treatise that dates back to *ca.* the 2nd century A.D., appears as a mere compendium of problems solved in a way that "dazzles rather than delights" and Diophantus himself is considered as someone looking only "for correct answers". This judgement comes from Kline [1972, p. 143] and is not an original one: some one hundred years earlier, in his *History of mathematics*, Cajori [1894, p. 77] had said: "Another great defect [of Diophantus' *Arithmetica*] is the absence of general methods". However, when Diophantus' methods are seen from their own historico-epistemological perspective — which requires, on the one hand, retracing their links to the Egyptian and Babylonian false position methods and, on the other hand, to the surveyors' geometrical cut-and-paste methods [cf. Høyrup, 1990] — Diophantus' methods appear to be very sophisticated. (A detailed study on this subject can be found in Radford, forthcoming.)

It seems implausible that (a) past mathematicians could have had a somewhat opaque vision of our modern concepts, and (b) that they could have been struggling, in their remote epoch, to bring their concepts as close as they could to our modern ones.

As soon as we leave the realm of the apriorist platonistic epistemology — and consequently, can no longer consider history as the account of the discoveries of the timeless Forms that were until then waiting to be discovered by fortunate humans, and, in turn, what counts is not solely the ideas that resemble the modern ones — we must make an

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explicit choice and say what it is that we want to see in the past. In other words, we must identify the interesting historical data.

Data is never interesting in itself. Historical data will always be interesting with regards to the conceptual framework upon which the research program relies. Furthermore, the framework makes it possible to offer a theoretical explanation of the data. In the case of the epistemological inquiries about the history of mathematics, the search for interesting historical data will be shaped particularly by our own perception of how mathematical knowledge grows[2]. We will only select to see what we *think* could have happened in the past. Thus, what a Vygotskian epistemologist will declare as "interesting" data, a radical constructivist may consider irrelevant. In other cases, both may pay some attention to the "same" historical phenomenon, but it will be read or interpreted differently. Of course, the frameworks do not need to be radically different for the "interpretation problem" to surface. Dubinsky noticed this problem when, commenting on a paper of Kaput and contrasting Kaput's results with Piaget-Garcia's, he said:

> This comparison shows how an interpretation of history can have at least as much to do with the interpreter as it does with any set of historical "facts". [Dubinsky, 1994. p. 158].

And added, promptly: " if indeed such things [the historical facts] exist." [op. cit., p. 158]

Another example of two different accounts of the "same" historical phenomenon is the following. The historical development of algebra has usually been divided into three stages: (a) rhetorical; (b) syncopated; and (c) symbolic. This division, introduced by Nesselmann more than one and a half centuries ago, is based on a very peculiar notion of mathematical abstraction that encapsulates one of the main ideas of *pure* mathematics. *Grosso modo*, this notion of abstraction is underlain by the idea that the representation of the object eliminates from the represented object all the "noise", "disturbance" and "dust" — an idea encompassed by a quasi-religious non-mathematical idea according to which the objects are purified by taking away all their insane physical substance. This is why "mathematics is seen to "elevate the mind" and get the "essence of the truth" " [Confrey, 1996, p. 40]. However, when the development of algebra is seen from a socio-cultural perspective, this division of algebra appears to be completely different: syncopated algebra was not an intermediate stage of maturation in which the knowledge took a kind of rest in its tiring race towards symbolism. Instead, it was a mere technical strategy that the limitations of writing and the lack of printing in past times imposed on the diligent scribes that had to copy manuscripts by hand. Indeed, many of the frequently used words were abbreviated by using their first letter. For instance, *c* for *cosa* (the thing), *r* for *radice* (the root) and *p* for *plus* (plus). Other words like "to multiply" were merely contracted. For instance, Luca Pacioli, in his printed *Summa de Arithmetica geometria Proportioni: et proportionalita.* [1523], still preserves the old tradition of hand-written texts, when he writes *m c a* for *multiplica* (multiply) [see Radford, 1996].

In order to go further in our discussion, let us suppose, for the sake of simplicity, that we make the fortunate epistemological choice concerning what to see in the past (a choice somewhat similar to the astronomer's when he chooses a certain point in the sky). Our focus is now well-directed, but just when we are finishing the calibration of our instrument and are finally going to see that much desired object, we realize that what we are "seeing" is necessarily mediated by our own *modern* social and cultural conceptions of the present *and the past*. Thus, wherever mediaeval historians placed their focus, they saw God's hand. The events of the past were seen as the accomplishment of *the divine plan*. Enlightenment historians, according to their scientific spirit, saw the past events as caused by rational, secularized forces, while Romantic historians, finding in the past "the expression of genuine and valuable human achievements" [Collingwood, 1994], became interested in investigating the past ages that the Enlightenment historians considered barbaric and without any historical value. The Romantic historians assumed a certain coherence *hidden* behind the phenomenal world and their task was that of grasping this coherence [Iggers, 1995].

Even the most titanic effort of putting away all our modern knowledge in order to see the historical event in its purity will not succeed: we are damned to bring our modern socio-cultural conceptions of the past with us. What is worse, it does not suffice to acknowledge the problem (as is often done) in order to solve it. Looking at the past in order to understand it is, as Gadamer [1975] suggested, a dialogical process in which two horizons (the past and the present) are fused together. There is no transparent lens [Confrey and Smith, 1994, p. 173]. The desired *contextualization* required in the mathematical historical inquiry, along with the "choc culturel" that such a contextualization provokes [Barbin, 1996] and the naivety praised by the ethnomathematical and anthropological studies are but manifestations of this dialogicality between two different worlds. Furthermore, this dialogicality is not restricted to the single objects under consideration: dialogicality is embedded in the dialoguing cultures and goes beyond any single object. Isolation, as an analytical methodological practice, is impossible. Belonging to and shared by its community, the object brings with it a lot more than *itself.* J.-P. Sartre alludes to this question when he said that the cultural presence of the 18th century, the austerity of Leipzig, the puritan coarseness of the German Princes are always there, given by the clear notes of Bach's Brandenburg concertos [Sartre, 1964].

This ontological scepticism has, it seems to me, two interesting corollaries for our discussion about the history of mathematics: (a) the "real" meaning of a past concept is unattainable; it will always be "filtered" by our framework and by our modern socio-cultural conceptions of the history, and (b) given that any historical investigation puts in contact two different horizons, and that the present horizon is always in movement, the history of any concept or of any theory will always be rewritten.

§2 Psychology and the history of mathematics

Let us now discuss a different problem that is specific to the teaching of mathematics, namely, the linking of historico-epistemological results to psychology and teaching.

Although, at a first glance, it seems evident that historical mathematical developments must have something to inform us about the difficulties that modern students encounter when they learn mathematics, a closer look at the situation reveals that it is far from easy to link both domains — the historical and the psychological. Again, the aprioristic platonistic epistemology displays a very simple and unsatisfactory solution — to *show* students how past mathematicians succeeded in *discovering* the mathematical concepts.

Waldegg[3] has suggested that the link between the history of mathematics and the psychology of mathematics can be ensured by epistemology. Still, we need to specify the kind of *link* that may be expected between both historical and psychological developments. As we said in the introduction, when this question is not ignored, very often one limits oneself to mentioning the psychological version of Ernst Haeckel's [1874] "law of recapitulation", according to which *ontogenesis* (i.e. the development of a subject's idea) recapitulates *phylogenesis* (i.e. the historical development of the idea)[4].

There are some constructivist and neo-constructivist works on mathematics education that refer (implicitly or explicitly) to the Piaget and Garcia view of the relationship between psychological and historical developments[5]. For example, when A. Sfard [1995] deals with the problem of the relationships between historical and psychological developments, she says:

> "Indeed, there are good reasons to expect that, when scrutinized, the phylogeny and ontogeny of mathematics will reveal more than marginal similarities. At least, this is what follows from the constructivist view according to which learning consists in the reconstruction of knowledge." [Sfard, 1995, p. 15].

and she justifies the alleged similarities between the phylogenesis and the ontogenesis by alluding to some inherent properties of knowledge that evoke the invariant mechanisms to which Piaget and Garcia refer in their "recapitulation" notion:

> "It is probably because of the inherent properties of knowledge itself, because of the nature of the relationship between its different levels, that similar recurrent phenomena can be traced throughout its historical development and its individual reconstruction." [Sfard, 1995, p. 15].

It is clear that the structural three-fold Piaget and Garcia idea is very appealing. However, recent developments in the sociology of knowledge are challenging recapitulationism [Wertsch, 1991] and the underlying critical pre-supposition of a social- and cultural-free cognizer whose knowledge may actually be decomposed into abstractive structural relations. Concerning this last point, in one of the most recent reflections on the socio-historical accounts of mathematics, M. Otte [1994, p. 309] says: "The development of knowledge does not take place within the framework of natural evolution but within the frameworks of sociocultural development" and "Knowledge is necessarily social knowledge." New developments in mathematics education are showing an increasing interest in taking into account social and cultural factors in the learning phenomena, which leads Wertsch and Toma [1995, p. 159] to say:

> "The claim that learning and development are inherently social is very much in the limelight these days. Instead of restricting our focus to the isolated individual when studying cognition and other forms of mental processes, we have come to realize that key aspects of mental functioning can be understood only by considering the social contexts in which they are embedded". Wertsch and Toma [1995, p. 159]

Concerning recapitulationism, almost 50 years ago Werner [1948] noticed that there are some irreducible differences in phylogenesis and ontogenesis that make them incomparable; among other things, he stressed the fact that the development of the child is encompassed by an interaction with the world of adults, and that primitive cultures are largely characterized by a vague organization of children until they arrive at adolescence when they are suddenly launched into the immutable social sphere of adults (something marked, for instance, by ritual ceremonies), while children from modern cultures are slowly led through a "plastic transformation from one stage of life into the other because of the intimate interdependence and interaction of the life patterns." [Werner, 1948, p. 27] Consequently, the modern child's mental activity recapitulates *neither* primitive child *nor* adult mental activity. According to Werner, what a phylogenesis-ontogenesis comparison may bring about is a merely *formal* connection [Op. cit. p. 28].

Being very interested in genetic analysis, Vygotsky himself considered the problem of the eventual relationships among different genetic domains. He suggested that the set of different principles governs each genetic domain making recapitulation impossible. He claimed, as Werner did, that in each case, the interaction between the socio-cultural history with the ontogenetic development of the culture under study makes its phylogenesis a particular and very specific event.

In fact, to understand conceptual developments we need to place the cognizer and the whole mathematical activity under study within his or her cultural conception of mathematics and of science in general. As Crombie has pointed out [Crombie, 1995, p. 232], the cultural conception of mathematics determines the organization of scientific inquiry, the kind of arguments that will be socially accepted, the kind of evidence and the type of explanations that will be considered valid. Indeed, the cultural conception of mathematics determines not only the social function of the mathematical knowledge but also — at a more abstract level — the conception of mathematical objects themselves.

I do not intend to say that the researchers in mathematics education who look at the past are insensitive to the existence of cultural factors. Indeed, very often, they do take into account that modern mathematical cognitive developments are embedded in modern teaching sequences which are, in turn, embedded in social and cultural contexts which are quite different from those of the past. One may even accept that the conditions of the actual psychological geneses of a mathematical concept are *ineluctably different* from their historical geneses. [cf. Artigue, 1990]. What I mean is that, when the question is explicitly addressed, one does not take the effects of such cultural differences seriously. Thus, for instance, Kaput himself [1994, p. 83] shies away from the crucial question by merely acknowledging, in an astonishing short phrase, that we have to be careful in using the recapitulation law for curriculum design purposes because of two reasons — one of them (but not the more important!) being the differences between "a collective historical enterprise and an individual's learning".

In their less radical manifestations, the mentalistic approach to knowledge assumes that cultural effects are always present but can be taken to be non-fundamentally significant. In contrast, the socio-cultural perspective suggests that the effects of culture and society are fundamental to the way in which we come to know. We shall return to this point in the following sections.

§3 Epistemological obstacles

Although the history of mathematics has been used in teaching contexts for at least a century[6], it was only some twenty years ago that the idea of analyzing mathematical knowledge from a historical perspective in order to shed some light on the students' processes of construction of knowledge came onto the scene. One of the pioneer works was that of G. Brousseau in the 70s, who transposed into mathematics G. Bachelard's notion of epistemological obstacle previously developed in his studies of scientific thought.

Bachelard [1938, p. 13], in a paragraph that has become famous, says:

> "Quand on cherche les conditions psychologiques des progrès de la science, on arrive bientôt à cette conviction que *c'est en termes d'obstacles qu'il faut poser le problème de la connaissance scientifique*. Et il ne s'agit pas de considérer des obstacles externes comme la complexité et la fugacité des phénomènes, ni d'incriminer la faiblesse des sens et de l'esprit humain: c'est dans l'acte même de connaître, intimement, qu'apparaissent, par une sorte de nécessité fonctionnelle, des lenteurs et des troubles. C'est là que nous montrerons les causes de stagnation et même de regression, c'est là que nous décèlerons des causes d'inertie que nous appelerons des obstacles épistémologiques."

The concept of epistemological obstacle gives Brousseau a way to interpret some of the recurrent and non-aleatorical mistakes that students make when they learn a specific topic. He claims that there is a logic behind these students' mistakes and explains them in terms of a knowledge that suffices to solve some problems fruitfully but fails to appropriately solve others. When applied beyond the limits of its scope or domain of validity, he says, the knowledge produces wrong results.

Given that the logic behind a recurrent mistake may have different causes, Brousseau [1983, p. 177] classifies the sources of obstacles as follows:

(1) an *ontogenetic source* (related to the students' own cognitive capacities, according to their development);

(2) a *didactic source* (related to the teaching choices);

(3) an *epistemological source* (related to the knowledge itself);

Epistemological obstacles (EO) are entailed by the third source and their detection will be made possible, according to Brousseau, through a confrontation of the history of mathematics and today's students' learning mistakes. Indeed, one of the roles of the didactician is, says Brousseau [1989, p. 42]:

(1) to find the students' recurrent mistakes and to identify the underlying conceptions,

(2) to find the obstacles in the history of mathematics and

(3) to compare the historical obstacles and the learning ones in order to determine their epistemological character.

As we can see, in their first formulations, epistemological obstacles were seen within the recapitulationistic parallelistic framework; they were seen as *an intrinsic difficulty of knowledge* [see Brousseau, 1983, p. 176] that makes unavoidable their reappearance in ontogenesis. Thus, for instance, Sierpinska [1985, pp. 7-8], after mentioning the controversies that the EO raised in didactic circles, says:

> "Pour notre part, nous retiendrons deux aspects de la notion d'obstacle épistémologique selon G. Bachelard (Bachelard, 1938):
>
> — l'apparition des obstacles a un caractère inévitable [...]
>
> — la répétition de leur apparition dans la philogénèse et l'ontogénèse des concepts."

She insists on the importance of the "unavoidable character" of epistemological obstacles some lines later when, referring to Bachelard's words (see above quotation), she says:

> "Ici, c'est le mot «nécessité» qui nous paraît le plus important. Il souligne le fait que cela n'a pas de sens de chercher à éviter les obstacles; on doit buter contre l'obstacle, en prendre conscience et ensuite le franchir pour progresser dans le développement de son savoir" [p. 8].

Although, in the early years of the EO perspective, everybody seemed to be aware of the problem that different cultural settings posed for the comparison of the mathematical knowledge of different times, the EO practitioners followed the recapitulationistic general strategy of minimizing socio-cultural effects that we put into evidence in §2. Thus, for instance, when Brousseau comments on Bachelard's list of obstacles, he says:

> Ces obstacles ont résisté longtemps. Il est probable qu'ils ont leur équivalent dans la pensée de l'enfant, bien que l'environnement matériel et culturel actuel ait sans doute un *peu* modifié les conditions dans lesquelles ceux-ci les rencontrent. [Brousseau, 1983, p. 173; our emphasis]

An important change is noticed in the development of the EO some years later, when the socio-cultural factors started to be considered less superficially. In his paper of 1989 Brousseau added the category of cultural obstacles to his first list — although, to my knowledge, the new category was not further elaborated, thus remaining at the level of a formal acknowledging sentence. At the same time, Sierpinska [1989] presented a revised epistemological obstacles program (in which she suggests that the recapitulationistic parallelistic hypothesis may be rejected) based on Wilder's chronologically well-ordered mechanistic view of mathematics as a cultural system where cultural agents interact between themselves like vectors are added in geometry! — an idea that runs parallel with Wilder's peculiar and curious view of the evolution of mathematics and which unfortunately reduces culture to *social behaviorism*, pretty much as in the case of Bloor's Strong Program [1976][7]. Sierpinska's revised program relies strongly on an identification of a cultural system as a system interrelated by three communicational links [that Wilder calls beliefs, mores and technology; Wilder, 1981, p. 12]: the formal (related, e.g., to the beliefs), the informal (containing e.g. the rules and "schémas inconscients de pensée"), and the technique (containing the explicit knowledge, logically justified, necessary in different professions) [see Sierpinska, 1989, p. 132]. She suggests that epistemological obstacles belong to the formal and informal level. We do not want to discuss here the problems that such a whimsical three-divisional view of a cultural system brings about when knowledge is examined through these levels. Let us only mention that it is far from easy to see what it is that keeps the technique level distant from the beliefs and mores of its own culture (and consequently from the epistemological obstacles themselves: the effects of culture may hardly be conceived as something that we can put *on or off* at pleasure!). We prefer to concentrate on the point related to our main discussion and submit (i) that by placing the EO in the formal and informal levels, the new account seems to contradict the original main conception of an epistemological obstacle — i.e. something intrinsic to knowledge — and (ii) that the new account leads us to the corollary that it is not possible to distinguish between cultural and epistemological obstacles. Further, within the new EO perspective, we might realize that epistemological obstacles are but cultural obstacles.

For instance, if we see the difficulties that western mediaeval mathematicians had in facing negative numbers, and if we see the difficulties encountered by our students today, we are led to think that, effectively, positive numbers constitute an obstacle to the emergence of negative numbers. However, if we retrace the negative numbers to Chinese mathematicians we see that they overcame the difficulty of handling negative numbers through a very clever representation using coloured rods (we shall return to this point in the next section). Thus, the difficulty that positive numbers pose to the rise of negative numbers is not an *intrinsic* problem of knowledge. It depends upon the *local*, cultural ideas about science, mathematics, their objects and methods.

The "sensitivity" of epistemological obstacles to cultural factors is acknowledged by Artigue [1995, p. 16],when she says that "epistemological obstacles identified in history are only candidates for obstacles in the present day learning processes". However, the examples she develops seem to illustrate the impossibility of recapitulationism rather than an irreproducibility of knowledge due to the intimate connection of this with its own, specific, and non-replicable culture, as this suggestive passage hints:

> As far as complex numbers are concerned, for instance, the situation of today's students for whom complex numbers are directly introduced as legitimate objects which are endowed from the start with punctual and vectorial geometrical representations, cannot be compared with the situation of Italian algebraists of the sixteenth century and even with that of their successors. [Artigue, 1995, p. 16]

This explains, I believe, the rather surprising question she raises immediately to the previous quotation: "What obstacles are resistant to these [cultural] differences?" [*Ibid*, p. 16]

No epistemological obstacle can "resist" the effect of culture for, if we are right, culture is not an inconvenience for knowledge nor does knowledge "fly" over cultures: as we shall attempt to show in the next section, knowledge is a cultural production ineluctably subsumed in its milieu.

§4 Mathematics and culture

We have been arguing that knowledge is deeply rooted in and shaped by its social and cultural context. In fact, a simple inspection of different cultures through history shows that each culture had its own scientific interests. Moreover, each culture had its own ways of defining and delimiting the form and the content of the objects of inquiry. In his enlightening work on the mediaeval (Muslim, Greek, Latin, Hebrew) encyclopaedias, McKeon showed how different they are and how these differences are underlain by intellectual factors specific to their corresponding cultures [McKeon, 1975]. In the case of mathematics, a good example is provided by the emergence of Greek deductive mathematics, which is often related to the political organization of the Greek city-states, based not on arbitrariness but on law, something that encouraged the citizens to argue and to debate. This is not entirely false, but it is also not entirely true. Beyond a causal, mechanistic, or behaviorist cultural reading of ancient mathematics, we might stress the fact that mathematics, as well as art and other symbolic accomplishments of any culture, are first of all semiotic manifestations of certain sensitivities of the culture that its members develop through shared experiences and from whence the meaning of the products are formed. As Otte has suggested, "symbolic systems represent crystallized co-operation" between the members of a culture[8]. It means that the Greek style of arguing and debating and its political climate are not merely concomitants to the way they conceived mathematics but are part of a large phenomenon that covers each activity of their culture. Thus, the Greek debating-arguing style formed part of a social intellectual trend consisting of a style of argumentation that flourished *ca.* the 5th century B.C. in some Greek philosophical circles. This style of argumentation was probably encouraged by the Eleatic opposition against the materialistic standpoint of the Pythagoreans. Indeed, we know that the Pythagorean materialism was challenged by a distinction between what can be grasped by sense-perception and by pure reason. Parmenides, in his well known poem *On Nature*, says that the former leads us to *opinions*, while the latter leads us to the *truth*. This distinction was a central point for Plato, who insisted that true knowledge (*episteme*) is the knowledge of the Forms and it cannot be achieved through sense.

In *The Republic* (602 d), Plato says:

> "A stick will look bent if you put it in the water, straight when you take it out, and deceptive differences of shading

> can make the same surface seem to the eye concave or convex; and our minds are clearly liable to all sorts of confusions of this kind."

Plato accommodated the insensible with the sensible by saying that the sensible phenomena (*aistheta*) participate in the Forms — the *eide*. Within this context, one of the problems was to explain the existence of things in terms of a few principles (e.g. wind and fire). Certainly, mathematical objects were seen as forming part of the insensible world attainable only through thought and organized through the general scheme of principles as we find in *Euclid's Elements* — an organization of mathematical objets that was compatible with the more general Greek idea of the world [see also Arsac, 1987].

Another example is provided by the emergence of the concept of equation in algebra. As we attempted to show in a previous work [Radford, 1996], the very early development of equations was related to the development of writing and to the socially accepted modes of mathematical explanation — modes that were shaped and shared by social structures. The appearance of equation as a mathematical object, something that happened during the Renaissance, required the emergence of another kind of rationality different from the practical one of the Maestri d'abaco [see Hughes, 1996, p. 215 ff].

Of course, one may retort that I am just speaking about the superficial form of mathematical practices and not about the very specific mathematical content hidden behind the superficial robes. The problem is that the elements I am considering are not superficial. They *are not* robes: they constitute the very nature of knowledge. Knowledge is not like an onion that we can peel or like a nut that we can crack to expose its kernel. Let me give one last example that illustrates this point very clearly.

E. Lizcano gives an interesting account of negative numbers in three different cultures — the classical Greek, the Alexandrian and the Chinese — and shows how the conception of negative numbers is profoundly related to the cultural prejudices, taboos, collective imageries and symbolisms that demarcate the ways to create different kinds of negativity. Thus, he says:

> "The use of sticks in [Chinese] calculations — something imported from non-mathematical practices, like the divinatory arts — incorporates in their mathematical manipulation some presuppositions and operational possibilities very different from those that transpose the alphabetical numerals or the numerical segments of Greek mathematics". [Lizcano, 1993, p. 62; our translation].

The presuppositions that Lizcano has in mind are:

(a) the sense of the black sticks (that represent the negativity of numbers) in the calculation board which permits them to have something that "would be an aberration for the Greek *episteme* — to make sticks disappear in order to construct emptiness" [*op. cit.* p. 62; our translation]; and

(b) the physical placement of the coloured sticks on the calculation board — something intimately linked to the spatial disposition and the structural configuration of the Chinese language — and whose existence will be charged with meanings taken from the culture. Thus, "the hole (*w u*) or Chinese's zero is a form of being with own particular rights within Chinese thought", that takes its vitality from the real core of the Chinese *episteme*: "opposition and equivalence", two elements that play analogous roles to the principles of identity and non-contradiction in the Greek *episteme*.

Lizcano gives a marvellous example of the opposition principle taken from a Chinese poem. Here are the two last verses:

> perfume lotus emerald water agitate breeze cool
> water agitate breeze cool summer day long
> long day summer cool breeze agitate water
> cool breeze agitate water emerald lotus perfume
> [See Cheng Chi-hsien, 1972, p.38; my translation]

Concerning the Alexandrian period, Lizcano stresses the fact that the decline of Greek rationality and the rise of neo-platonism creates a fracture in classical thought that makes it possible to think of things from new perspectives.

Indeed, it is at this very moment that the philosophical neo-platonic speculations about figural numbers of the early Pythagoreans re-appeared. Furthermore, it is at this moment that we not only see the Pythagorean speculations reborn but also a sharp interest in previous great civilizations (e.g. Egyptian, Babylonian). For instance, Iamblichus (who wrote a commentary about Nichomachus' *Introduction to Arithmetic*[9]) wrote his book *The Mysteries of Egypt*[10].

This decline, says Lizcano, opens up the possibility of thinking about some of the *irrationalisms* and *unwise* things in the classical Greek *episteme* — beliefs, habits; but also, in the realm of mathematics, to dare what, until then, had been unthinkable: to (a) rethink the meaning of the monad or "One"; (b) to perceive a certain negativity of numbers, as we find in Diophantus' *Arithmetica*, and to proclaim, as Nichomacus does in his *Introduction to the Arithmetica*, the primacy of arithmetic over geometry.

That which is most important for our discussion is that the new forms and contents of thinking are not merely concomitants of their society and culture. In fact, Diophantus' concept of number, like his own multicomposite Alexandrian culture, is impregnated with the influence of classical Greek thought (particularly by the works of Euclid and Aristotle), and the heritage of Egyptians and Babylonians. Hence, although he gives the classical Aristotelian definition of numbers as "formed by a certain quantity of units" [Ver Eeck (ed.), 1926, p. 1], he considers the monad (the number "one" — one of the most fundamental problems of Greek thinking[11]) as *any* other number. He even dares to operate with it, something that Euclid meticulously avoids — to the point of giving two different proofs for essentially the same theorem [see Radford, 1995]. On the other hand, the monad (like the Egyptian's number one) becomes divisible and the fractions are considered as authentic numbers in the diophantine calculations. The fact that Diophantus is not restricted to the Aristotelian definition of number (which confers a new vigour to his calculations) and that his numbers have an ambivalent identity is, in the light of Lizcano's analysis, due to a relaxation of traditional rationality caused by the aforementioned cultural fracture that characterizes the end of Antiquity — a fracture that permits Diophantus to give up one of the two pillars of the classical Greek *episteme*: the principle of identity.

The previous examples clearly suggest, I believe, one of the main ideas that we have been trying to elaborate: that mathematical knowledge is *more* than merely concomitant with its cultural environment and that the configuration and the content of mathematical knowledge is properly and intimately defined by the culture in which it develops and in which it is subsumed. Consequently, any attempt to study it might take into account the composite extra-mathematical cultural structure in which mathematical knowledge is embedded.

§5 Concluding remarks

In this paper I have pointed out some different problems that surface when the history of mathematics is used as a non-naïve tool in mathematics education.

All these discussions raise a question: what, then, could be the role of historico-epistemological analysis for the researcher in mathematics education?

I think that if, as the socio-cultural perspective suggests, knowledge is a process whose product is obtained through negotiation of meaning which results from the social activity of individuals and is encompassed by the cultural framework in which the individuals are embedded, the history of mathematics has a lot to offer to the epistemology of mathematics. Indeed, historico-epistemological analyses may provide us with interesting information about the development of mathematical knowledge within a culture and across different cultures and provide us with information about the way in which the meanings arose and changed; we need to understand the negotiations and the cultural conceptions that underlie these meanings. The way in which an ancient idea was forged may help us to find old meanings that, through an adaptive didactic work, may probably be redesigned and made compatible with modern curricula in the context of the elaboration of teaching sequences [for two examples related to algebra, see Radford and Grenier, forthcoming and Radford and Guerette, 1996]. As Cantoral [1995, p. 57] pointed out, we need to recognize the complexity of a modern theory and to historico-epistemologically unravel it, in order to reconstruct accessible presentations for our students.

A cultural historico-epistemological investigation may inform us about the way in which competitive research programs confronted each other at a certain moment in the development of mathematics and to better understand the issues of such confrontations, seeing the confrontations not only through the cognitive lenses of the victorious programs but also within the context of the sociocultural values and commitments at stake in such confrontations [see Glas, 1993].We will have to keep in mind, of course, that an ancient problem or an ancient mathematical situation will never again be the same. It seems that Heraclitus was right when, standing on the river bank and looking at the flow of water, he said that it is not possible to step into the same river twice.

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Notes

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2 To better understand the difference between the epistemological task we have in mind and the historical one, we may take Unguru's description of the history:

> "History is primarily, essentially interested in the event qua particular event ... History is not (or is primarily not) striving to bunch events together..." [Unguru, 1994, p. 208].

In contrast, the link between events, with regard to the formation and the growth of knowledge, is certainly the task of epistemology.

3 See Waldegg, G. [1997] Histoire, épistémologie et méthodologie dans la recherche en didactique, *For the Learning of Mathematics*, this issue.

4 Transposing his biological law of recapitulation to psychology, Haeckel said:

> "Mais c'est tout à fait de la même manière que l'admirable activité intellectuelle de l'homme est sortie graduellement, à travers des milliers d'années, de la grossière intelligence des vertébrés inférieurs, et le développement psychique de chaque enfant n'est qu'une brève répétition de cette évolution phylogénétique". [Quoted by Mengal, 1993, p. 94]

5 As is very well-known, Piaget and Garcia suggested that there is a relationship between psychological and historical developments and that this relationship must be seen not in terms of *content* but in terms of *mechanisms* mediating the transitions from one stage to another with the three stages through which an idea supposedly undergoes in its development being: (a) the intraoperational, (b) the interoperational, and (c) the transoperational; each one of these stages being characterized by the way in which objects are seen and related to themselves [Piaget and Garcia, 1989]. The intentional absence of the word recapitulation in their work may be seen as an effort to stay away from the naive idea of a recapitulationism of contents.

6 Rogers [1995, p. 108] appropriately characterizes Cajori's book, *A History of Elementary Mathematics with Hints on Methods of Teaching*, published in 1896, as a text providing an inductivist account of the history of mathematics and hinting at a teaching that mirrors the 'inductivist' progress of this science.

7 "Then we can conceive of the evolution of mathematics as an ordered array of vectorial systems in which the vectors grow in magnitude at varying rates." [Wilder, 1981, p. 16]

8 Otte, M. Mathematics, Semiotics, and the Growth of Social Knowledge, *For the Learning of Mathematics,* this issue.

9 Iamblichus (ca. 300) *In Nicomachi Geraseni Arithmeticam introductonem*, Samuele Tennulio (ed.), 1668 (Microfilms of the University of Toronto).

10 Iamblichus (ca. 300) *Les mystères d'Égypte,* Paris: Les Belles Lettres, 1991.

11 The association of the monad with the beginnings of things may be retraced in many Greek texts. For instance, in the *Theologomena Arithmeticae* — a book belonging to the Pythagorean tradition and often attributed to Iamblichus, we read: "The monad is the non-spatial source of number ... Everything has been organized by the monad, because it contains everything potentially" Iamblichus, [tr. Waterfield, 1988]

References

Arsac, G. [1987] L'origine de la démonstration: essai d'épistémologie didactique, *Recherches en Didactique des Mathématiques*, Vol. 8, No. 3, pp. 267-312

Artigue, M. [1990] Épistémologie et Didactique, *Recherches en Didactique des Mathématiques*, Vol. 10, No. 2, 3, pp. 241-286. Artigue, M. [1995] The role of epistemology in the analysis of teaching/learning relationships in mathematics education, *Proceedings of the 1995 Annual Meeting of the Canadian Mathematics Education Study Group*, Y. M. Pothier (ed.), University of Western Ontario, pp. 7-21

Bachelard, G. [1938] *La formation de l'esprit scientifique*, Paris: Librairie philosophique Vrin, réimpression, 1986

Barbin, E. [1996] *Histoire et Enseignement des mathématiques: pourquoi? Comment?* Conférence donnée au Centre Interdisciplinaire de Recherche sur l'Apprentissage et le Développement en Éducation, CIRADE, Univerité du Québec à Montréal, juin 1996

Bloor, D. [1976] *Knowledge and Social Imagery*, London, Henley and Boston: Routledge & Kegan Paul

Brousseau, G. [1983] Les obstacles épistémologiques et les problèmes en mathématiques, *Recherches en Didactique des Mathématiques,* Vol. 4, No. 2, pp. 165-198

Brousseau, G. [1989] Les obstacles épistémologiques et la didactique des mathématiques, in: *Construction des savoirs, obstacles et conflits*, N. Bednarz et C. Garnier eds., Montréal: les éditions Agence d'Arc inc., pp. 41-64

Cajori, F. [1894] *A history of Mathematics*, New York: Macmillan and Co.

Cantoral, R. [1995] Acerca de las contribuciones actuales de una didáctica de antaño: el caso de la serie de Taylor, *Mathesis*, Vol. 11, No. 1, pp. 55-101

Cheng Chi-hsien [1972] Analyse du langage poétique dans la poésie chinoise classique, *Tel Quel*, Vol. 48-49, pp. 33-46

Collingwood, R. G. [1994] *The Idea of History*, Oxford/NewYork: Oxford University Press, Revised Edition

Confrey, J. [1996] A Theory of Intellectual Development, part III, *For the Learning of Mathematics*, Vol. 15, No. 2, pp. 36-45

Confrey, J. and Smith, E. [1996] Comments on James Kaput's chapter, in: *Mathematical Thinking and Problem-Solving*, A. H. Schoenfeld (ed.), Hillsdale, New Jersey / Hove, UK: Lawrence Erlbaum Associates, pp. 172-192

Crombie, A. C. [1995] Commitments and Styles of European Scientific Thinking, *History of Sciences*, Vol. 33, pp. 225-238

Dubinsky, E. [1994)] Comments on James Kaput's Chapter, in: Mathematical Thinking and Problem-Solving, A. H. Schoenfeld (ed.), Hillsdale, New Jersey / Hove, UK: Lawrence Erlbaum Associates, pp. 157-171

Gadamer, H.-G. [1975] *Truth and Method*, New York: Crossroad, Second Revised Edition, 1989

Glas, E. [1993] Mathematical Progress: Between Reason and Society, *Journal for General Philosophy of Sciences*, Part I: 24, 43-62. Part II: 24, 235-256

Høyrup, J. [1990] Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought, *Altorientalische Forschungen*,17, 27-69, 262-354

Hughes, B. [1996] Arabic Algebra, Victim of Religious and Intellectual Animus, in: *Mathematische Probleme im Mittelalter*, M. Folkerts (ed.), Wiesbaden: Harrassowitz, pp. 197-220

Iggers, G. [1995] Historicism: The History and Meaning of the Term, *Journal of the History of Ideas*, Vol. 56, N° 1, pp. 129-152

Kaput, J. [1994] Democratizing Access to Calculus: New Routes to Old Roots, in: *Mathematical Thinking and Problem-Solving*, A. H. Schoenfeld (ed.), Hillsdale, New Jersey / Hove, UK: Lawrence Erlbaum Associates, pp. 77-156

Kline, M. [1972] *Mathematical Thought. From Ancient to Modern Times,* Oxford University Press, Vol. 1

Lizcano, E. [1993] *Imaginario colectivo y creación matemática*, Barcelona: editorial Gedisa

McKeon, R. [1975] The Organization of Sciences and the Relations of Cultures in the Twelfth and Thirteenth Centuries, in: The Cultural Context of Medieval Learning, J. E. Murdoch and E. D. Sylla (eds.), pp. 151-192

Mengal, P. [1993] Psychologie et loi de récapitulation, in: *Histoire du concept de récapitulation*, P.Mengar (ed.), Paris, Milan, Barcelone, Bonn: Masson, pp. 93-109

Otte, M. [1994] Historiographical Trends in the Social History of Mathematics and Science, in: *Trends in the Historiography of Sciences*, K. Gavroglu *et al.* (eds.), Kluwer Academic Publishers, pp. 295-315

Pacioli, L. [1523] *Summa de Arithmetica geometria Proportioni: et proportionalita*. Novamente impressa. Toscolano: Paganinus de Paganino.

Piaget, J and Garcia, R. (1989) *Psychogenesis and the history of science*, New York: Columbia University Press

Radford, L. [1995] La transformación de una teoría matemática: el caso de los Números Poligonales, Mathesis, Vol. 11, No. 3., pp. 217-250

Radford, L. [1996] An Historical Incursion into the Hidden Side of the Early Development of Equations, In: *Arithmetics and Algebra Education*, J. Gimenez, R. Campos Lins and B. Gómez (eds.), Tarragona, Spain: Universitat Rovira I Virgili, pp. 120-131

Radford, L. (forthcoming) Elementary Algebraic Thinking from the Perspective of the Didactic Epistemology, in: Algebraic Processes and Structure, R. Sutherland, T. Rojano, A. Bell and R. Lins (eds.).

Radford, L. and Guérette, G. [1996] Quadratic equations: Re-inventing the formula. A teaching sequence based on the historical development of algebra, In: *Proceedings of the Quadrennial Meeting of the International Study Group on the Relations Between History and Pedagogy of Mathematics and Deuxième Université d'été Européenne sur l'Histoire et l'Épistémologie dans l'éducation Mathématique*, Universidade do Minho, Braga, Portugal.Vol. II, pp. 301-308

Radford, L., Grenier, M. [forthcoming] Entre les idées, les choses et les symboles. Une séquence d'enseignement d'introduction à l'algèbre, *Revue des sciences de l'éducation*

Rogers, L. [1995] The Historical Construction of Mathematical Knowledge, *Actes de la première université d'été européenne. Histoire et épistémologie dans l'éducation mathématique*, 105-114, F. Lalande, F. Jaboeuf (éds.), IREM de Montpellier

Sartre, J.-P. [1964] Situations IV, Paris: Éditions Gallimard.

Sfard, A. [1995] The Development of Algebra: Confronting Historical and Psychological Perspectives, *Journal of Mathematical Behavior*, Vol. 14, pp. 15-39

Sierpinska, A. [1985] Obstacles épistémologiqes relatifs à la notion de limite, *Recherches en Didactique des Mathématiques*, Vol. 6, No. 1, pp. 5-67

Sierpinska, A. [1989] Sur un programme de recherche lié à la notion d'obstacle épistémologique, in: *Construction des savoirs*, N. Bednarz et C. Garnier (eds.), Ottawa: Agence d'Arc Inc., pp. 130-147

Unguru, S. [1994] Is Mathematics Ahistorical? An Attempt to An Answer Motivated by Greek Mathematics, in: *Trends in the Historiography of Sciences,* K. Gavroglu et al. (eds.), Kluwer Academic Publishers, pp. 203-219

Ver Eecke, P. [1926] *Diopante d'Alexandrie. Les six Livres arithmétiques et le Livre des nombres polygones*. Desclée de Brouwer. Liège. Reprinted: Albert Blanchard, Paris, 1959

Waterfield, R. (tr.) [1988] Iamblichus, Theologomena Arithmeticae, Phanes Press.

Werner, H. [1948] *Comparative Psychology of Mental Development.* New York: International Universities Press, Inc., Second Printing, 1957.

Wertsch, J. V. [1991] *Voices of the Mind. A Sociocultural Approach to Mediate Action,* Cambridge, Ma.: Harvard University Press.

Wertsch, J. and Toma, C. [1995] Discourse and Learning in the Classroom: A Sociocultural Approach, in: *Constructivism in Education*, L. P. Steffe and J. Gale (eds.), Hillsdale, New Jersey / Huve, UK: Lawrence Erlbaum Associates

Wilder, R. L. [1981] *Mathematics as a cultural system*, Oxford, New York, Toronto, etc.: Pergamon Press