SIMILARITY AND EQUALITY IN GREEK MATHEMATICS: SEMIOTICS, HISTORY OF MATHEMATICS AND MATHEMATICS EDUCATION

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In this article I want to say a few words about equality and similarity in Greek mathematics. I wish to highlight, among other things, how the formula "equal and similar" reflects the distinct character of "equal" and "similar" as signs in Greek mathematical discourse. Beyond this, however - and, perhaps, chiefly - I want to claim that a discussion of this type is relevant not only to history of mathematics but to mathematics education as well

But why should its relevance be questioned in the first place? Obviously, the notions of similarity and equality, or rather congruence, remain central in every geometry curriculum History, it would seem, should only heighten what is already being covered in the classroom The question arises because of *the kind of understanding* we have today of similarity and equality and, therefore, the kind of emphasis a classroom treatment typically requires. There are differences in the specific approaches teachers adopt for teaching similarity and equality, but the general trend is to view them in terms of transformations and invariants, even if this is done implicitly. Certainly this is the case in dynamic geometry environments (*e.g.*, King & Schattschneider, 1997; Lehrer & Chazan, 1998) The attraction and power of transformations is not hard to see: framing similarity and congruence in terms of transformations often allows one to prove geometrical facts simply and elegantly (*e.g.*, Yaglom, 1962) and certainly allows one to treat geometry in a unified way, as Felix Klein made clear at the end of the 19th century. But these modern ideas and the view of similarity and congruence they contain are, as we shall soon see, very different, even *radically* different, from the view of the Greeks.

Of course one can *force* Greek conceptions into this modern mold - indeed it *has* been done - but this means abandoning history or stretching it, anyway, on a procrustean bed of modernity So it looks like one must either give up history or give up relevance. Precisely this situation was behind the argument in Fried (2001) that the attempt to bring history of mathematics into mathematics education creates a dilemma for both. The question of relevance remains, then, very pressing indeed

Before semiotic considerations entered the scene, it would have been difficult indeed to answer this question of how a discussion of Greek ideas on similarity and equality might be relevant to *modern* mathematics education and difficult to escape the dilemma of mathematics education and history of mathematics. It is still difficult, but the semiotic outlook provides at least an indication of where the answer might lie Semiotics has brought to mathematics education the awareness that mathematical meanings are not separable from the signs that carry them and that the signs themselves are products of human activity. But these are very nearly the basic presuppositions of the history of ideas; so, to the extent that semiotic ideas are relevant to education, historical issues (not mere matters of chronology) become educational issues.

The article begins with a discussion of equality and similarity in Greek mathematics I show that congruence or superposition was the fundamental expression of the idea of equality and remained the chief measure of equality in Greek geometry Similarity, however, had no basic measure nor a basic formulation except, somehow, sameness of shape Finally, I show how there was no real continuity between equality and similarity, and that that lack of continuity created the sense for the phrase "equal and similar" (*isos te kai homoios* in Greek). Throughout I tacitly claim that "equal" and "similar" are signs and that they give rise to other signs, such as "equal and similar." In the second part, I set out these semiotic assumptions more explicitly and show that, from a semiotic perspective, history of mathematics indeed becomes an essential part of mathematics education, rather than a mere tool for it

Equality and similarity in Euclid and Apollonius

Equality

The ancient theory of equality is contained in the five common notions (*koinai ennoiai*) at the start of Book I of Euclid's *Elements*. These common notions are as follows:

1. Things equal to the same thing are equal to one another.
2. And if equal things be adjoined (*prostethēi*) to the same thing, the wholes are equal
3. And if equals be removed from the same thing, the remainders are equal.

4 And things fitting (*epharmozonta*) on one another are equal to one another.

5 And the whole is greater than the part. (vol. 1, pp 5–6) [1]

Of these, the fourth provides the basic criterion, the basic test, for equality: if one figure can be fitted exactly on top of another, then the figures are equal (*isos*). This criterion appears also in Book VI of Apollonius' *Conica*, whose first definition is:

> Conic sections which are called equal are those which can be fitted, one on another, so that the one does not exceed the other. Those which are said to be unequal are those for which that is not so. (p. 264) [2]

Apollonius makes use of the definition immediately in the first proposition of Book VI where he proves that "Parabolas in which the *latera recta* which are the parameters of the perpendiculars [ordinates] to the axes are equal are themselves equal, and if parabolas are equal, their *latera recta* are equal" (p. 266) [3] In the demonstration itself one sees that the phrase "fitted one on another" is meant quite literally. Although proposition VI.1 refers to a metrical aspect of the parabola – namely, the length of the *latus rectum* – its proof applies the definition directly, therefore keeping the concrete geometrical object itself, the parabola, always in view.

In the *Elements*, the fourth common notion makes its first appearance in proposition I.4, which states that two triangles are equal if they have equal sides surrounding an equal angle. The entire demonstration, problematic though it may be, plays on common notion 4: equal lines are fitted on equal lines; the equal angle is fitted on the equal angle; this forces the one triangle to fit on the other, so that the two are equal. In fact, that equal lines can be fitted on equal lines and equal angles on equal angles is not truly an application of common notion 4, but its converse. However, with simple undivided objects like lines and angles, being equal and fitting on one another are identical relations (see Fried & Unguru, 2001, p. 228). [4] That said, we should not pass over the matter so quickly. It is an important fact that the common notion states only the sufficiency of "fitting on one another" for equality, for in *Elements* I.35, Euclid shows that things can be equal without having the same shape, that is, without one being able to fit on the other.

Elements I.35 states that "Parallelograms on the same base and in the same parallels are equal to one another" (Euclid, vol. 1, p. 48). Thus the proposition tells us that parallelograms such as ΑΔΓΒ and ΕΒΓΖ (see Figure 1), which are clearly not the same shape, may, nevertheless be equal.

In his translation of Euclid, Heath (1908/1956) remarks that "we are in this proposition introduced for the first time to a new conception of equality between figures" (vol. 1, p. 327). But this, at very least, is misleading. To understand why it is necessary to recall how Euclid proves I.35:

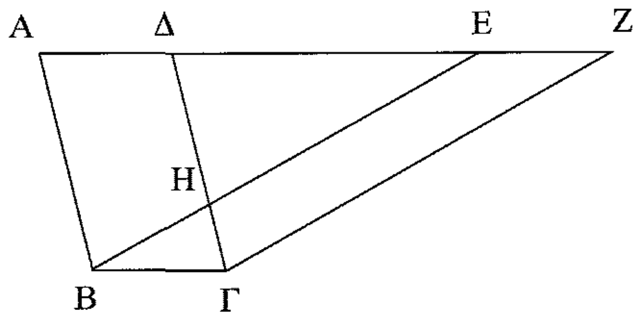

Figure 1: Parallelograms on the same base and in the same parallels are equal to one another.

> For since ΑΔΓΒ is a parallelogram, ΑΔ is equal to ΒΓ. For the same [reason], ΕΖ is equal to ΒΓ. Whence also ΑΔ is equal ΕΖ [common notion 1]; and ΔΕ is common; therefore, the whole ΑΕ is equal to the whole ΔΖ [*common notion* 2]. But also ΑΒ is equal to ΔΓ. So the two [lines] ΕΑ, ΑΒ are equal to the two [lines] ΖΔ, ΔΓ, each to each. Also the angle contained by ΖΔΓ is equal to the angle contained by ΕΑΒ, the exterior to the interior. Therefore, the base ΕΒ is equal to the base ΖΓ, ΔΓ and the triangle ΕΑΒ is equal to the triangle ΔΖΓ [*I.4 and derivatively, therefore, common notion 4*]; let the common [part] ΔΗΕ have been removed; therefore, the remaining trapezoid ΑΒΗΔ is equal to the remaining trapezoid ΕΗΓΖ [*common notion 3*]; let the common triangle ΗΒΓ have been adjoined; the whole parallelogram ΑΒΓΔ is therefore equal to the whole parallelogram ΕΒΓΖ [*common notion 2*]. The parallelograms being on the same base and in the same parallels are, therefore, equal to one another. *Hoper edei deixai* [QED] (vol. 1, pp. 48–49)

What we see in this proposition is no "new conception of equality" but how, beginning *from* the basic criterion of "fitting on one another," the common notions – what I have called the theory of equality – imply that equality applies also to shapes that do not fit on one another. It is an elaboration of equality, a spelling out of the theory, not a new kind of equality, certainly not one based on a metrical notion of area. Indeed, it can be argued that all of Books I and II are an elaboration of what "equality" means, including *Elements*, I.47, the "Pythagorean Theorem," which shows not only how one figure can equal another of a different shape, but also how one figure can equal two others of the same shape.

In sum, the *Elements* presents a broad notion of equality whose full extent lies in the possibilities allowed by the five common notions. It is not restricted to the case where one figure fits on another, which we call congruence; but, it is rooted in this very geometrical idea. This is a crucial point: proposition I.35 does not introduce a new idea of equality, equality through equality of area; rather, the areas are equal *because* the figures are equal, and the figures are equal because they are generated by shapes that fit on another.

Similarity

Like equality, the foundation of similarity rests in the *look* of geometrical figures. I use the word *look* intentionally; it is a fair translation of the Greek word *eidos*, which in the context of Plato's works is commonly translated as "form." In mathematical contexts *eidos* could as well be translated as "shape," but "form" *in the sense* of "shape." For example, in Euclid's *Data*, we have the following definition (Definition 3): "Rectilineal figures are said to be *given in form* (*tōi eidei*), if the angles are given one by one and the ratios of the sides to one another are given" (*Data*, after Taisbak, 2003, p. 17). [5] This definition from the *Data* is also notable not only because of its resemblance to the definition of similarity in the *Elements* (see below), but also because it makes clear that "similar" is related to something that the figures themselves possess, [6] namely, their look or form: to be similar (*homoios*), then, is to be in possession of the same shape, the same form, the same look

But what *is* the shape of something? That depends on what the "something" is. Unlike equality, which possesses one basic criterion, there are as many criteria for similar figures as there are different kinds of shapes For the Greek mathematician, there is no single governing criterion for similarity. In the context of the *Elements*, for example, we have the following various criteria for similarity:

> *Elements III (definition 11), Similar circular segments*: "Similar segments of circles are those which contain equal angles or in which there are angles equal to one another." (I, p. 93)
>
> *Elements VI (definition 1), Similar rectilineal figures*: "Similar rectilineal figures are such that they have each of their angles equal and sides about the equal angles proportional." (II, p. 39)
>
> *Elements XI (definition 9), Similar solid figures*: "Similar solid figures are those contained by similar plane areas (*epipedōn*) equal in number." (IV, p. 2)
>
> *Elements XI (definition 24), Similar cones and cylinders*: "Similar cones and cylinders are those of which the axes and diameters of the bases are proportional." (IV, p. 3)

Add to these, other definitions from Apollonius and Archimedes:

> *Apollonius, Conica VI (definition 2), Similar conic sections*: "... similar [conic sections] are such that, when ordinates are drawn in them to fall on the axes, the ratios of the ordinates to the lengths they cut off from the axes from the vertex of the section are equal to one another, while the ratios to each other of the portions which the ordinates cut off from the axes are equal ratios." (p. 264) [7]
>
> *Archimedes, Conoids and Spheroids (Introduction), Similar obtuse-angled conoids (i.e. hyperbolas of revolution)*: "Obtuse-angled conoids are called similar when the cones containing the conoids are similar." (Archimedes, p. 154)

Each of these definitions – and they are, I stress, definitions – demands prior knowledge of the object to which the word "similar" is being applied; for example, one must know that the angles contained in a segment of a circle are all equal so that one can speak of *the* angle of the segment. The necessity of such prerequisite knowledge precludes a general definition of similarity: "similar" always awaits the particular geometrical entities that are to be similar. And the things that are to be similar cannot be any things To ask whether or not a triangle is similar to an ellipse may be, for us, a pedantic question, but it is an askable question; for the Greek mathematician, by contrast, it would be like asking whether *War and Peace* is as long as a trip from New York to Paris: "similar" for triangles is one thing and "similar" for conic sections something else.

Equal and similar

The particular meanings carried by "equal" and "similar" in Greek mathematics induces yet another sign "equal and similar" (*isos te kai homoios*); this, conversely, further clarifies "equal" and "similar" themselves. As Mugler (1958) noted, that as a unit of meaning, as a sign, there is an earlier non-mathematical usage of *isos te kai homoios*, as when Thucydides describes how Corinth declared a colony for Epidamnus where there would be for all those desiring to go "... perfect equality" (*epi tei isei kai homoiai*) In mathematical contexts, "equal and similar" points to situations where the converse common notion 4 of the *Elements* holds. But to see "equal and similar" as merely fulfilling a logical function would, I believe, miss the essence of the matter: there is "equal," there is "similar," and there is "equal and similar." Thus, in the *Elements*, the definition of "equal and similar" solid figures appears in Book XI *as an independent* definition (Defn. 10) after that of similar solid figures (quoted above): "Equal and similar solid figures are those contained by similar planes equal in multitude and in magnitude" (in Heath's translation, vol. 3, p. 261) Euclid might have defined equal and similar solid figures as similar solid figures whose faces are not only similar but equal as well, that is, he might have subordinated "equal and similar solids" to "similar solids," and, therefore, also eliminated the need for a separate definition to start with. Obviously there is a relationship between "similar solids" and "equal and similar solids," but in the Euclidean discourse it is a relationship between denizens of distinct categories

This understanding of "equal and similar" can also be seen in Book VI of Apollonius' *Conica,* which is wholly dedicated to similarity and equality of conic sections. Proposition VI.16, for example, tells us that "Opposite sections are similar and equal" (p. 304) [8] Proposition VI.2 already shows that opposite sections are equal, by which, again, Apollonius means the curves may be fit on one another; yet, Apollonius will not say the opposite sections are similar until he has a specific criterion for similarity of hyperbolas – and for that he has to wait until VI.12 Therefore, to be *equal,* even when this means able to be fit one on another, and to be *similar* are different relations for Apollonius. The essential distinction between "equality" and "similarity" makes the "equal and similar" a necessary third sign The string of words "equal and similar" exists for us as well, of course, but not as a distinct sign: for us, to say "similar and equal" is understandable, but it is a redundancy, just as saying "rectangular and square" is.

In sum, Euclid's and Apollonius' use of "equal" and "similar" is always directed towards concrete geometrical objects, never to the relations that characterize those objects, though such relations may provide criteria for equality and similarity. It is this focus that makes "equality" and "similarity" in Greek mathematics quite different, indeed, radically different, from our own "equality" and "similarity." For us, these refer to transformations, and the key idea of a transformation is that it acts not on objects but on the space containing them. Hence, similarity, in that view, must be definable *before* it is referred to any particular geometrical object. That Greek geometers did *not* view similarity and equality in this way made their explorations of equality and similarity explorations of the subtleties of geometrical shapes. Without taking that into account, one seriously misunderstands what mathematicians such as Apollonius were trying to do. Thus, by assuming the modern point of view to be Apollonius' point of view, Zeuthen (1839–1920), for example, whose expansive and deep 1886 work on the *Conica* set the tone of Apollonius scholarship for almost a century, could conclude summarily that Book VI of the *Conica* on equality and similarity, "is of no great significance" (p. 384) in the attempt to understand the Greek theory of conic sections.

Semiotic considerations

Zeuthen's remark on Apollonius' book on equality and similarity of conic sections, Book VI of the *Conica*, brings us back to the query, raised in the introduction: why should investigating Greek mathematics be relevant for *modern* mathematics education? Zeuthen was a mathematician who believed that a modern mathematician was best equipped to understand the history of mathematics, in particular, the mathematics of Apollonius and Euclid. [9] He could judge Book VI of Apollonius' *Conica* to be "of no great importance" for the understanding of Greek mathematics since he viewed Greek mathematics only as modern mathematics in an ancient language: what is trivial from a modern mathematician's point of view, in his view, could not therefore have been of great interest from a Greek mathematician's point of view. Approaching the history of mathematics as Zeuthen did – and this is also how many mathematics educators have approached the history of mathematics – one is likely either to find the history of mathematics trivial or to discern in it only significant *modern* mathematics, a little like a historical version of what Brousseau called the "Jourdain Effect," after the character in Molière's play (see Brousseau, 1997, pp. 25-26).

But what I hope I have shown by examining the significations of "equal," "similar," and "equal and similar" in Greek mathematics is that one cannot assume that modern mathematics and ancient mathematics had the same intentions, even when they used the same words. Indeed, it should be clear from the discussion above that only by looking carefully at words and other signs in Greek mathematics one can begin to grasp what Apollonius' and Euclid's mathematical understanding really was.

Yet, may it not be that in showing Greek mathematical understanding to be different and distant from ours, one has only proven the *irrelevancy* of Greek ideas of "equality" and "similarity" for *modern* mathematics education? I think not. My reasons, as I related in the introduction, have to do with the constellation of emphases in modern mathematics education that are embraced within the field of semiotics: mathematical meaning and understanding, social and cultural mediation, and communication. Semiotics has come to mean many things to many people, but, as it relates to mathematics education, its main insights, put in very broad terms, are: 1) that there is no firm division between meaning and understanding and communication; 2) that these are all governed by transactions involving signs – words, symbols, images, gestures; 3) that interpretation, along with problem-solving, calculating, and proving, is an essential activity in mathematics learning and thinking. These insights suggest that mathematical texts, including student productions and discourse *viewed as texts,* are central objects of study for mathematics education. Since the study of the history of mathematics involves reading mathematical texts closely, we have here some justification for a discussion such as that above on "similarity" and "equality." But of course *some* justification is not enough: why one should study *historical* mathematical texts is still unanswered.

In considering the specific question of history of mathematics and mathematics education, the semiotic ideas stemming from Ferdinand de Saussure are particularly useful. Saussure's thought proceeds from linguistics, in particular that brand of linguistics based in late 19th century historical philology. Linguistics and historical philology being what they are, Saussure's semiotics, or semiology as he called it, was set from the start in a cultural-historical context. Ironically, Saussure's *Course in General Linguistics,* where his semiotic ideas were set out, was an attempt to break away from that historical tradition; yet, it was in that very attempt that the place of culture and history in semiotics truly came to light. The key Saussurean distinction, in this context, is that between "diachronic" and "synchronic" linguistics. Diachronic linguistics considers signs in terms of their evolution in time, for example that the Latin *amicus* evolved into the French *ami*. From the diachronic perspective, signs are the way they are in a given language because they have *become* that way over time; language, from this perspective, is dynamic and, in a way, capricious: "... diachronic events have always an accidental and particular character" (Saussure, 1916/1974, p. 131). [10] Synchronic linguistics considers signs as part of a *system* of signs; in Saussure's words, it is concerned "... with the logical and psychological relationships among coexisting and system-forming terms, as they are perceived by the same collective consciousness" (p. 140). Viewed synchronically, signs are the way they are because of their connection with or, more precisely, *their difference from* all other signs in the system. Language, from the synchronic perspective, is a single static structure all of whose signs have a definite and immutable place.

It is tempting to identify diachronic linguistics with the historical philology which Saussure was rejecting. But this is not so. The 19th century historical philologists assumed that an historical account was sufficient and, therefore, could identify historical philology as linguistics itself. Diachronic linguistics, it is true, is historical; however, Saussure's claim is that, far from providing sufficient means for understanding language, diachronic linguistics cannot explain the structural aspects of language, which is the domain of synchronic linguistics But this is not to say that synchronic linguistics is in any position to explain how one sign system changes into another, which is the domain of diachronic linguistics. The two views of linguistics run orthogonal to each another. They are at once incompatible and complementary, like the cross-section and longitudinal section of a plant, to use one of Saussure's images (p. 125). [11] Although the two viewpoints cannot be adopted simultaneously, they also cannot exist without one another, and, in that sense, they define one another. This makes the synchronic viewpoint, in its own way, a historical one: a language viewed synchronically must be viewed as a distinct sign system existing at a particular moment in history [12] For this reason, intellectual history may be conceived as an effort to grasp synchronies of the past (see Fried, 2004)

How this plays out in the educational development of a mathematical idea can be derived from our purely historical discussion of "similarity" and "equality" in the last section Pursued in an educational setting, the point of such a discussion, even without all the details, would not be to scratch and find in Euclid's *Elements* exactly what we do in our own classrooms And when we have recognized differences between our treatment of "similarity" and Euclid or Apollonius' - for example, that geometrical equality is not, for them, a special case of similarity - the point should not be to underscore what we know and what Euclid or Apollonius did not: for the difference between the modern view, in which equality is only a special case of similarity, and the ancient view, in which equality and similarity are related though distinct enough to require a distinct treatments, is not a difference built on a different set of facts but a different set of meanings. The sign "equal and similar," especially in Apollonius' case where equality of conic sections is exclusively "fitting on one another," does not reflect Greek *ignorance* of transformations. "Similar" and "equal" have a meaning for Euclid and Apollonius that makes "equal and similar" a necessary third sign for them. Moreover, that these signs have a meaning for them, for *both* Euclid and Apollonius, is part of what it means to have meaning, namely, its commonality in a communicating community [13] Saussure says early on in the *Cours* that a language system exists identically for all members of a speaking community, or in our case, a mathematical community. Saussure (1916/1974) puts it this way:

> The language system (*la langue*) exists in the collective as a sum of imprints set in every mind, a little like a dictionary, identical copies of which are distributed among all individuals. It is thus something which is in each individual, something common to all but outside anyone's will. This mode of existence of the language system may be represented by the formula:
>
> 1 + 1 + 1 + 1 + ... = I (collective model) (p. 38)

That sum of imprints, that commonality of a collective, is not far from what we have in mind when we utter the word "culture " In this way, studying synchronies of the past is studying past cultures and establishing a distinct relationship with the past.

And here, Saussure's distinction between the synchronic and diachronic viewpoints becomes important, for it brings out the paradoxical nature of that relationship: we see the origins of our own signs in those of the past, as we see *amicus* in *ami*, yet how our own signs fit together, how they form a system for us, how ultimately we understand them, cannot alone explain the sign systems, the synchronies, of another time, and conversely. There is, in this sense, a fundamental, unavoidable tension in our confrontation with the past that our educational approach must be cognizant of and must embrace. In effect, this tension is precisely that in our confrontation with difference itself - and, as is clear from the discussion above, the idea of difference is central to all of Saussure's thought.

Educationally, then, in studying equality and similarity in Greek mathematics, we study a *different* mathematical culture, and, we thus learn to see mathematics as an expression of culture. We sense that different culture is connected to ours and yet is also somehow unapproachable This is the tension that comes with our engagement with difference, and in it we gain a broader understanding of how our own mathematics is truly distinct and what it is within our own culture. For students, this means seeing how such notions as similarity are, in modern mathematics, set in a numerical or functional semantic domain; number and measure color most of our geometrical notions, not only similarity and equality, but also ratio, angle, and area, among others. A historical approach informed by the considerations discussed in this section ought, therefore, to sharpen our way of treating and thinking about similarity and equality, precisely because it is not framed as a mere translation of the Greek view.

Conclusion

In a paper on the relevance of semiotics in mathematics education, Radford wrote:

> ... since signs denote objects, semiotics is urging us to better clarify the very nature of the mathematical objects with which we deal in our classroom practices The point is not necessarily to end up philosophizing in the mathematics classroom (although it would do no harm, I guess) but to better understand our own practices and the cognitive, epistemological and education role of the semiotic systems that we are encouraging in the classroom. [14]

Understanding our own mathematical practices, understanding our own mathematics altogether, entails understanding ourselves working within a system of signs. But if we never wander away from that system, we shall be caught in the vicious cycle of trying to understand our own sign system with our own signs.

Naturally, we can never completely free ourselves from our own signs, but we can try to become aware of them by engaging in a different system of signs Such an engagement is close to Bakhtin's idea of a dialogic encounter And about that he says: "Such a dialogic encounter of two cultures does not result in merging or mixing. Each retains its own unity and *open* totality, but they are mutually enriched" (Bakhtin, 1986, p. 7). The message is one that has been heard often in the last half century, almost to the point of being hackneyed, namely, that we find out about ourselves through our engagement with the other. But this is precisely the reason we study history; Collingwood thus spoke of "history as the self-knowledge of the mind" (Collingwood, 1939) And, again, it is self-knowledge of the mind not because our thought is identical with that of the past, not because our thought mirrors the past, but because its nature is as something ever in the process of breaking away from the past. It is both inescapably defined by and other than the past.

As educators, our job then should not be to use the history of mathematics merely as a ploy to catch students' attention; rather, it should aim to *position* students with respect to the past, to bring them to see themselves as beings poised with one foot in the past Of course, what we are depends on where both feet are; however, that place cannot be fully grasped – and this has been my point throughout this article – without squarely facing the ground beneath our foot in history. In this way, by honestly confronting, for example, Euclid's thought about "similarity" and "equality," that is, by letting historical mathematical texts speak in their own voice, students take a firm step towards understanding their own voices, their own distinct mathematical selves.

Notes

[1] Unless otherwise noted, translations of Euclid (based on Heiberg's edition, 1969-1977) are mine A reference to Euclid's text, such as "vol 1, p 3," means page 3 of the first volume of Heiberg's edition. Heiberg mentions other common notions appearing in some manuscripts – *e.g* "If unequals be adjoined to unequals, the whole is unequal" and "Doubles of the same thing are equal to one another," as well as, "Two lines do not contain a space" (vol I, pp. 5-6) Except for the last about two lines not containing a space – which, as Proclus himself seems to suggest, is a red herring among the others – all of these common notions, therefore, concern equality

[2] Unless noted otherwise, all quotations from Book VI of the *Conica* are translated by Toomer (1990).

[3] See Fried & Unguru (2001)

[4] This is true also for Apollonius proposition cited above, particularly in the second part; however, Apollonius defines the equality of conic sections such that they are equal *if and only if* they coincide

[5] Taisbak's book contains a deep discussion of the idea of "given-ness in form" and of Euclid's *Data* generally.

[6] In this connection, Taisbak (2003) points out, "*The Elements compare triangles. The Data deals with individuals,* and with the 'knowledge' we may have of such individual triangles, within the language of Givens" (pp 126-127, emphasis in original)

[7] See Fried & Unguru (2001).

[8] The "opposite sections" are what we call the two branches of the hyperbola, or, simply, the hyperbola. Apollonius, by contrast, views them a pair of hyperbolas For a discussion of the "opposite sections," as opposed to the "hyperbola," see Fried (2004).

[9] For a review of Zeuthen's historiographic assumptions and style., see Lützen & Purkert (1994)

[10] All translations from Saussure are mine.

[11] In Fried (2004), the incompatibility of the diachronic and synchronic aspects of a sign system was emphasized to bring out the fundamental difficulty in using history of mathematics in mathematics education – in particular, where mathematics education is concerned with the inculcation of modern mathematical concepts and procedures

[12] Otherwise, one could explain why French is the way it is by explaining why Latin was the way it was, obviating the need for the diachronic viewpoint altogether Conversely, if one wants to understand Latin as a language, one must see it in terms of its own set of immutable relationships among its own signs, despite their similarity to signs in modern French.

[13] Probably slightly less than a hundred years separated Euclid and Apollonius, and mathematics was surely not static during those years Nevertheless, we may still include them in a single mathematical community, in the sense that they shared a set of mathematical meanings Saussure referring to language, emphasizes that it is the degree to which a community shares a language system that makes them one community. Hence, one can legitimately speak about historical periods: "In practice, a state of the language system *(langue)* is not a point, but an interval of time which can be long or short as long as the accumulated changes are minimal It can be ten years, a generation, a century, or even more" (Saussure, 1916/1974, p. 142)

[14] On p 10 of Radford, L (2001) 'On the relevance of semiotics in mathematics education', paper presented to the Discussion Group on Semiotics and Mathematics Education at the 25th Conference of the International Group for the Psychology of Mathematics Education, University of Utrecht, Utrecht, NL

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