TALKING WITH THE LITERATURE ON EPISTEMOLOGICAL OBSTACLES

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This discussion focuses on the process of studying for my doctoral degree and the thoughts that came to mind. While the investigation itself concentrated on epistemological obstacles, I was investigating the epistemological obstacles that mathematics students at undergraduate level encounter in coming to understand the limit concept in the context of a function and a sequence, pursuing the study was also full of epistemological obstacles for me

Acquiring the concept epistemological obstacle

Having been a mathematics teacher in some Lesotho high schools and now a mathematics lecturer at the National University of Lesotho (NUL), I have always been interested in teaching the 'slow' learners. Because of this, I decided to pursue a study that would help me understand the problems and difficulties that learners encounter in learning I chose the title of my study to be *Obstacles that mathematics students at undergraduate level encounter in understanding the limit concept*. There is ample evidence from the literature that students encounter problems in learning calculus and, in my case, this was evidenced by the high failure rate in calculus at the National University of Lesotho.

The operational definition that I had suggested for the term 'obstacle' was *anything that hinders the students' progress in learning*. With this conception in mind, I handed in the first draft of my proposal to my supervisors One comment they made took me a long time to settle mentally,

> This is very broad. There are a number of obstacles that may hinder students' progress in learning. Which obstacles do you want to investigate?

It was not easy for me to answer this question. When searching the literature I came across the work of Cornu. I read his chapter on limits where he introduces a number of obstacles:

> genetic and psychological obstacles which occur as a result of the personal development of the student, didactical obstacles which occur because of the nature of the teaching and the teacher, and epistemological obstacles which occur because of the nature of the mathematical concepts themselves. (1991, p 158)

From reading this passage, I was then aware that there are indeed a number of obstacles so I had to choose one type of obstacle as a matter of focus. Because of my passion for wanting to confront the causes of students' problems in learning, I thought that the notion of cognitive obstacle would be the one to investigate But the idea that the notion of epistemological obstacle is related to the nature of the mathematical concepts never left my mind. When we talk about the nature of the mathematical concepts, what are we actually talking about? Cornu has given the following as examples of the epistemological obstacles of the past:

- the failure to link geometry with numbers
- the notion of the infinitely large and infinitely small
- the metaphysical aspect of the notion of limit, and
- is the limit attained or not? (1991, pp. 159-161)

Since I was not used to this terminology, I did not see how this list constituted the nature of the limit concept In particular, I was puzzled by the question, "[I]s the limit attained or not?"

I came across the work of Sierpinska (1987) where she investigated the notion of epistemological obstacle with humanities students. Here, Sierpinska talks about the dual nature of epistemological obstacles She categorises epistemological obstacles into *heuristic* and *rigouristic*. Only a diagram was used in illustrating these obstacles, which I found difficult to understand However, in accessing the more detailed work to which Sierpinska had referred the reader, I found another problem - it was written in French, as is the case with the work of Bachelard.

In her work, Sierpinska has written about four notions which she says are the sources of epistemological obstacles related to limits: scientific knowledge, infinity, function and real number. She used these notions as her framework for the questions she set for humanities students I compared these ideas to the epistemological obstacles listed by Cornu. What was common between them was that they seemed to be the ideas that are encountered when dealing with limits and therefore could be said to constitute the nature of the limit concept. These ideas seem to be unavoidable when learning or discussing the limit concept I had made a decision - I needed to investigate epistemological obstacles. This does not mean that I had a better understanding of epistemological obstacles. I changed my topic of study to *Epistemological obstacles that mathematics students at undergraduate level encounter in coming to understand the limit concept*.

My next task was to clarify my notion of epistemological obstacle. Sierpinska writes:

> We know things in a certain way. But the moment we discover there is something wrong with this knowledge (i.e become aware of an epistemological obstacle), we understand something and we start knowing in a new way. This new way of knowing, in its turn, start[s] functioning as an epistemological obstacle in a different

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situation. Not all, perhaps, but some acts of understanding are acts of overcoming epistemological obstacles. And some acts of understanding may turn out to be acts of acquiring new epistemological obstacles

> A description of the acts of understanding a mathematical concept would thus contain a list of the epistemological obstacles related to the concept, providing us with fuller information about its meaning.
>
> In many cases overcoming an epistemological obstacle and understanding are just two ways of speaking about the same thing. The first is "negative" and the other "positive". Everything depends upon the point of view of the observer. Epistemological obstacles look backwards, focusing attention on what was wrong, insufficient, in our way of knowing. (1990, p. 28)

I then struggled with reconciling the interpretations by Cornu and Sierpinska. Epistemological obstacles as obstacles related to the nature of the subject matter and epistemological obstacles as a wrong, insufficient way of knowing, said to be negative Brousseau gave the following explanations of epistemological obstacles:

> Obstacles of really epistemological origin are those from which one neither can nor should escape, because to their formative role in the knowledge being sought. (1997, p. 87)

The question that remained for a long time was: How can a wrong way of knowing play an informative role in the knowledge to be acquired?

Herscovics used the work of Bachelard as the base for his explanations and did the translation from French himself:

> When one looks for the psychological conditions of scientific progress, one is soon convinced that it is in terms of obstacles that the problem of scientific knowledge must be raised. The question here is not that of considering external obstacles, such as the complexity and transience of phenomena, or to incriminate the weakness of the senses and of human spirit; it is in the very act of knowing, intimately, that sluggishness and confusion occur by the kind of functional necessity. It is there that we will point out causes of stagnation and even regression; it is there that we will reveal causes of inertia which we will call epistemological obstacles. (1989, p. 61)

I felt at ease in using this translation because I could refer to its source. In this quotation, Bachelard talks about epistemological obstacles as the "causes of stagnation" in the knowledge to be acquired. He also says that this stagnation occurs as a "functional necessity". To me, this seemed to be a positive way of looking at an epistemological obstacle. Examples of epistemological obstacles that Herscovics had found from the work of Bachelard were:

- the tendency to rely on deceptive intuitive experiences
- the tendency to generalize, and
- the obstacles caused by natural language.

By looking closely at these epistemological obstacles, I also found them to be unavoidable situations For example, mathematics shares some of its technical terms with the natural language we use in everyday life. So, when we get into the classroom situation, we cannot in anyway avoid retrieving some of this knowledge Also, before higher education, where logical deductions are used as methods of proof, we still use our intuitions. Sometimes they are right, sometimes they are wrong, but since they are among our available sources of knowledge we rely on them to a certain extent Reading the historical development of the limit concept confirmed the idea of functional necessity I therefore started looking at these obstacles as positive because they are the stepping stones to the knowledge to be acquired

What remained a problem for me from the work of Herscovics was that he associates the term 'epistemological obstacle' with the past and he prefers to use the term 'cognitive obstacle' in the present. This is because he refers to epistemological obstacles as obstacles that were encountered in the development of the scientific knowledge and cognitive obstacles as the obstacles related to individual learning This perception created a lot of mental conflict for me. My problem was: if these epistemological obstacles are the causes of stagnation in the knowledge to be acquired, does it matter by whom and when? I referred back to the work of Cornu, Sierpinska and Brousseau. These authors show that these obstacles appear only in part in history, they are also found in educational practice today. Because this view resonated with my understanding of knowledge acquisition, I settled for it

Acquiring the concept, can a function attain its limit?

This question appeared in the history of the limit concept and was initially asked as "Can a variable attain the limit value?" As I had studied calculus only through interaction with mathematics text books, I experienced this question for the first time in reviewing the literature. The ideas that came to mind during this encounter were: What does it mean to attain the limit value? Is it the same as saying "Will the function values equal the limit value?" Does it mean the same thing as reach the limit value?

In the work of Taback, the question asked in relation to the word 'reach' was:

> A rabbit starts at one endpoint, say A, of a line segment AB. On his first hop, the rabbit jumps halfway from A to B On his second hop, the rabbit jumps halfway from where he is toward point B The rabbit continues to hop, following the same rule of correspondence: every time he takes a hop, he jumps halfway from wherever he is toward point B. Does the rabbit reach point B? (1975, p. 111)

In reacting to the stated problem, Taback says:

> The answer to the question depends upon the interpretation of the word "reach". The rabbit will certainly not reach Point B, in the sense of landing on B, after a finite number of hops The mathematician however, says that the rabbit will reach point B, meaning that the rabbit's hops converge to B as a limit point; that is,

> the rabbit can get and remain within any given neighbourhood around B. (1975, p. 111)

The conception of "reach" being in the neighbourhood of a point stayed with me for a long period of time. Any reflection on this meaning brought a comparison with the everyday meaning of the word. In some cases, we say we have reached our destinations when we have landed on them. Sometimes we say we have reached some points when we are in their neighbourhood. But this did not in any way answer my question whether or not 'reach' and 'attain' are synonymous.

Tall (1991) explicitly talks about whether or not the limit can be attained in relation to the *generic extension principle*. An example given being that the convergent sequence $1/n$ tends to the limit zero, but the terms never actually equal zero. Thus, the terms of the sequence cannot attain the limit value. And such an observation was made by looking at the terms of the sequence through the limiting process of 'tending to'.

With this conception in mind, I experienced a mental conflict at a later stage when reading the work of Juter ([1], 2005), since:

- the equality of the function value and the limit value was not obtained through the limiting process of 'tending to' but by considering any function value that was equal to the limit value, and
- the words 'reach' and 'attain' were used synonymously – an interpretation that is different from that of Taback.

For instance, Juter's subjects were asked whether or not the function could attain the limit value in

$$\lim_{x \to \infty} \frac{x^5}{2^x}$$

One of the responses from the subjects that was considered to be right was, "Yes, for $x = 0 \longrightarrow f(0) = 0/1 = 0$". This example was different from that of Tall. The function here was said to attain the limit value without the application of the limiting process of 'tending to' but by considering a case in which any function value would be equal to the limit value regardless of position. Here the domain process is, $x \longrightarrow \infty$, something very far from choosing 0 in the substitution.

With regard to the interpretation of the two terms 'reach' and 'attain', this is what Juter has written:

> The question whether limits are attainable seems to be confusing (Cornu, 1991; Williams, 1991). All the students' responses to questions and tasks about it presented in this paper are incoherent. This study shows that the students interpret the definition as stating that the limits cannot be reached by the function. When they solve problems on the other hand, they can see that sometimes limits are attainable for functions ([1], p. 42)

My problem was still with the interpretation of the term 'reach' from the work of Juter and Taback. Tall had not used the word 'reach' but 'attain'. Another of my confusions was how the function can or cannot attain the limit value. Is it by a consideration of a limiting process as in the case of Tall? Is it by an observation of any function value that is equal to the limit value as the work of Juter suggests? With the knowledge that the limit value is obtained through a coordinated process schema, $f(x) \longrightarrow L$ as $x \longrightarrow a$, that is, in evaluating the function $f(x)$ values of points in the neighbourhood of a are considered but with each successive point being closer to a than the previous point, I therefore had to settle for the conception of attain from the work of Tall. As Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas and Vidakovic (1996) observe, this process involves an infinite number of computations. Some computations will be performed while others will just be contemplated, as there are infinitely many points to be considered.

Epistemological obstacles and concept acquisition

So, I encountered "Is the limit attained or not?" as an epistemological obstacle in coming to understand the limit concept, because I could not give a proper interpretation for the word 'attain'. I could also not make a clear distinction between the meanings of the words 'attain' and 'reach'. This problem was compounded when I encountered different interpretations of these words by researchers who constitute the mathematical community. The assumption that I made was that, since the context is the same, the sense in which these terms are used would also the same. Cornu suggested that the word 'reach', when used by mathematicians, meant being in the neighbourhood of a point and I shared this view. This experience, however, has taught me that this is not necessarily the case. Within the same context meanings have to be negotiated. When a word is used, the important question to ask would be, 'In what sense is this word being used?' For example, is 'attain', in the mathematical context, used in the sense of Juter, Taback or Tall? All three researchers agree that sometimes the limit values are attainable for functions, what differs is the sense in which they use the word 'attain'. There is a need for clarification of meaning in communication regardless of context

Sierpinska has described an epistemological obstacle as a wrong, inadequate way of knowing. When I encountered the meanings of the word 'attain' by different researchers it seemed difficult for me to point out with certainty that a particular interpretation is a wrong way of knowing as my choice of a certain type of interpretation originated from my own sources of conviction

In the work of Bachelard, epistemological obstacles as causes of sluggishness and confusion are said to occur as a functional necessity in knowledge acquisition. This description seems to have been the general trend in the way in which I encountered epistemological obstacles in my concept acquisition. On a number of occasions in discussing my experiences in the preceding sections, I have explicitly used the word confusion as a description of how I felt when encountering epistemological obstacles. As I encountered more epistemological obstacles in coming to understand the limit concept my understanding was enhanced by reflecting on these obstacles, rejecting some, and keeping some as part of the process of the reconstruction of the new schema. Sierpinska talks about understanding and overcoming epistemological obstacles as one and the same thing. This is the understanding that I came to acknowledge through my experiences.

The experience of working on mathematical understanding in the context of a limit as a researcher has made me realise that the teaching of mathematics and the investigation of mathematical understanding are complementary processes As a mathematics teacher, my focus was more on procedures or methods that have to be applied in solving mathematical tasks to achieve correct answers. As a researcher, my understanding of what constitutes teaching mathematics now goes beyond paying attention to methods or procedures used in obtaining the correct answers. In my teaching, I now see myself as having a dual role, that is, being a teacher and a researcher.

Notes

[1] Juter, K. (2003) *Learning limits of functions, university students development during a basic course in mathematics*, and *Learning limits of functions: students solving tasks*, unpublished Licentiate thesis, Luleå, Sweden, Luleå University of Technology, Department of Mathematics

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[These notes and references follow on from page 26 of the article "Talking about order of operations" that starts on page 25 (ed.)]

- the critical role of symbols and models is emphasized, for example, real objects or a graphic model of a part-whole relationship assists children to see all component quantities and their relationship at the beginning of the curriculum

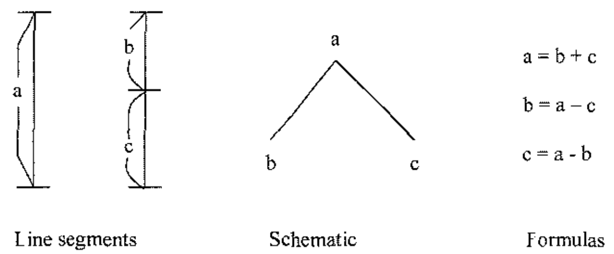

Figure 5: The part-whole relationship

In fact, this early understanding of the part-whole relationship played a critical role in children's interpretation of the order of operations described in this article Figure 5 shows a variety of graphic models and formulas that illustrate the part-whole relationship.

Later, in word problems/text activities, children analyze the text using various methods such as a line segment, a schematic, or a formula. At the same time, children make up texts/questions focusing on the analysis of the relationships

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