The Number-Line Metaphor in the Discourse of a Textbook Series

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The following is an account of the role of the number-line metaphor in an Argentine textbook series (Tapia, Bibiloni and Tapia, 1974, 1975). The development and work of the metaphor is tracked from its emergence as a representation of the natural numbers to the introduction of the real numbers. It is part of a study of textbook construction of the real number system which investigated a way to analyze texts used in the teaching of mathematics. [1]

Most of the research on mathematics textbooks can be considered as external critiques of text. Metaphorically speaking, textbooks have been conceived as technological products (as in Howson, 1995), containers (as in van Dormolen, 1986), or funnels (as in Kang, 1990) of the mathematics to be learned. The analysis of textbooks has yielded descriptions and explanations that refer them to the educational system, the mathematics of the mathematician, or the process of didactic transposition. [2]

In contrast, Otte (1983) has analyzed the interaction between diagrams and explanations as an emergent of the conflict between the temporal and the spatial nature of texts He has also explained how metaphors in textbooks may be more than pedagogical devices: metaphors can also be textual strategies which collaborate in the construction of the meaning that an item of knowledge has in the text. His approach can be described as conceiving the textbook as an environment for the construction of knowledge and yields descriptions that could be called internal critiques.

The present study, close to Otte's approach, investigated a way to carry out an internal critique of a text that would allow one to address questions like: how does textuality as a linear temporal process create its own mathematics? And what are the textual meanings of the mathematical notions which are developed inside a textbook and along that temporal axis?

The main theoretical influences on my efforts to conceptualize and analyze text have been Foucault's (1972) archaeology of knowledge, Chevallard's (1991) theory of didactic transposition and Eco's (1979) notions of model reader and open-closed work. However, their presence is mostly silent; my understanding of their works guides my way of interrogating and asserting, but they are not responsible for my actual questions and conclusions.

The textbooks analyzed were written for the first and second year of high-school mathematics [3], and are part of a series that also includes books for the third and fourth years [4] A modern mathematics flavor is apparent in these books, which set off to develop arithmetic and geometry as different topics in the same language of the 'theory of sets' (see Howson *et al.*, 1981, pp 100-101). Owing to the explicit intent to present mathematical knowledge as organized by its community of reference, these textbooks seemed to present a nice challenge to the claim that a text unfolds a distinct regime of mathematical discourse Thus, I chose to study them even though they were published over 20 years ago.

The number line: a metaphor?

Tapia *et al* (1974, Ch. 4-6, 8) present the natural numbers as cardinals of sets and derive the natural number operations from operations with sets. The Euclidean line, which had been introduced to the reader within an axiomatic development of plane geometry, is recalled to produce a representation of the natural numbers, 'the number line':

> The mathematician often uses graphic language to clarify or interpret some concepts So, for example, the representation of the set N on a line allows us to visualize some properties of the set N. On the line R, one marks a point o and chooses a segment U as a unit. The segment U is translated consecutively from o To each point of division one matches sequentially a natural number. [5]

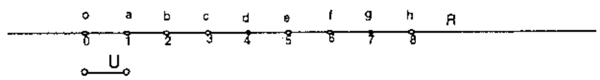

Figure 1
The number line for the natural numbers (pp 114-5)

I take this as the presentation of the line as a metaphor for the number system.

Tapia *et al.* (1974, 1975) put together the subjects of arithmetic and geometry in the same volumes, but such an integration rarely reaches the level of the chapter and never that of the section. The common thread that unifies Tapia *et al* (1974) is the notion of sets, but the number sets and the sets of points of the plane officially emerge and develop independently. On the one hand, the successive enlargements of the notion of number (which form the discourse on numbers of Tapia *et al*, 1974) largely follow the genetic approach of defining an equivalence relation and taking the quotient set The discourse of geometry, on the other hand, attempts an axiomatic development of plane Euclidean geometry where the figure, as a prominent theme, is produced by the aggregation of points Therefore, though they are related by the set-theoretic language and some set-theoretic notions, the discourses on numbers and of geometry can be distinguished in terms of the *objects* they deal with and the *strategies* for the production of such objects (see Foucault, 1972, pp. 64-70)

As a result, one reason to call the number line a metaphor is the observation that the line does not belong to the canonical discourse about numbers. On the one hand, *line* is officially introduced in Chapter 2 as a primitive concept for geometry. On the other hand, the invocation of *line* to produce the number-line representation is subordinated to the strategies of geometrical discourse, notably the number-line representation is introduced only after the axiom of order of the line (stated in Tapia *et al.*, 1974, p. 114, in a chapter on equality and order in N) can guarantee that all natural numbers will find a representative on the line. In that sense, introducing the number line into the discourse about numbers is *focusing* on (natural) numbers inside the *frame* of the Euclidean line (to use Black's (1962) characterization of metaphor as *focus-inside-a-frame* (p. 28).

Now one can ask: how different are the phrases 'the number line' and 'the perpendicular line'? Clearly, the second phrase seems to address a commonplace about lines (words like *perpendicular*, *parallel*, *tangent* and *intersecting* are associated with *line* in the canonical discourse of geometry). The first phrase, however, is not commonplace in geometric uses of line: a line has points, not numbers, and usually those points are labelled with letters, not with numbers At face value, the phrase 'number line' is a *catachresis* in the sense of "the application of a term to a thing which it does not properly denote" (Oxford English Dictionary, quoted in Black, 1962, p. 33).

As in the case of Pimm's (1987) 'spherical triangle', the 'number line' in Tapia *et al.* (1974) comes from a basic analogy

> number : number system :: point : line

The viability of the representation of the numbers on the line, and thus the textual emergence of *the number line* as a catachresis in the sense of the "putting of new senses into old words" (Black, 1962, p. 33), are based on the above analogy. Its later metaphorical action rests on that basic viability

Pimm (1987) asks:

> What is the force of calling *spherical triangles* triangles? Such geometric configurations of curves ('lines') on spheres may not have had a descriptive term previously. Classifying them as triangles results in stressing the function of this configuration (that is, three segments of great circles meeting pairwise in three points) in the study of the geometry of the sphere, likening it to the role played by the concept of triangle in the plane. Immediately whole theories, comprising definitions, concepts and theorems line up for examination, 'translation' and exploration. (p. 102)

A second reason to treat the number line as a metaphor is a working hypothesis Could the notion of metaphor – particularly Max Black's (1962, 1979) notion of metaphor, which Pimm (1987) has specialized for the case of mathematics – illuminate the work of the number line in the discourse about numbers? Tapia *et al.* (1975) eventually conclude:

> To every real number corresponds a point on the line. To every point on the line corresponds a real number. There is a bijective function that maps the set of real numbers onto the set of points of the line. (p. 164)

Consider Pimm's (1987) conjecture on the spherical triangle: "In the process, the adjective *spherical* becomes as appropriate or acceptable as is *equilateral* as a classifier for a type of triangle" (p. 102) Could the number line in Tapia *et al* (1974, 1975) be a case in which the role of metaphor in the production of mathematical propositions could be documented? In other words, would the initial presence of the line as a pedagogical device in the discourse about numbers have any textual influence in the production of the bijection between the Euclidean line and the set of real numbers as a mathematical proposition?

The elaboration of the metaphor

Tapia *et al.* (1974) introduce the number line on page 114, drawing on it to display some properties of the set of natural numbers, in particular its inductive infinity and its discreteness In Chapter 6, they define the operations of addition and subtraction with natural numbers and provide a 'number-line interpretation':

> Take, for example, 3 + 4 = 7 One counts 3 units from zero and then 4 units to the right of the point 3 The point that one obtains represents the sum Generalizing: to represent the sum $a + n = b$, one counts n places to the right of the point a. Remember: 'to add 4' means "to count 4 units to the right". 'To add n' means "to count n units to the right"

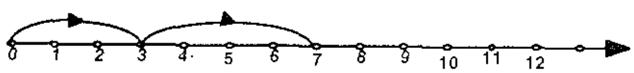

Figure 2
Addition of natural numbers (p 129)

What is at stake, however, is not how much 3 + 4 is, but rather what the structural properties of addition of natural numbers are (and how a number-line interpretation of addition would clarify these properties). The verbal description of the number-line figure for addition regards the first addend as the initial position for a transformation and the sum as the final position after a transformation Yet, the figure provides another feature: addition on the number line erases that context by identifying each addend and the result with arrows Hence, number-line addition, although described as an asymmetric counting process, is shown as a symmetric display of a numerically stated addition.

The process of representing addition on the number line provides a bridge between the set-theoretic notion of addition and the practices of addition already known by the students: the number line keeps the independence of addition (achieved in the set-theoretic construction) from the base of the numeration system, and provides an environment where actual additions can be performed (i.e., by counting). However, an important achievement of the display of number-line operations is to start developing a number-line register by producing a term-by-term correspondence between statements of addition in the arithmetic register and number-line figures The two examples below illustrate how this development reinforces the metaphor.

Additive practices on the number line

The statements involving addition on the number line ask the reader not to perform but to interpret additions on the line. What is there for the reader to do with the number line? The statement of an addition on the number line involves the juxtaposition of two arrows, each indicating a relative position. Such a graphic restatement of the additive problem makes the result evident and only requires one to decide the starting and ending points of the third arrow. Hence, the number line does not leave a gap between the statement of an additive problem and its solution. The achievement of the solution to the additive problem is part of the competence required to state the problem in number-line terms There is a problem for the reader to deal with, but it is not an additive problem: the remaining problem is to parse and translate addition statements from the arithmetic to the number-line register back and forth.

In fact, the exercises for number-line addition and subtraction are of two kinds: "Interpret the following additions on the number line" (p. 134) and "Write the sums represented on the number line" (p. 134). In the rest of the addition or subtraction exercises, the number line is never mentioned Hence, the role of the number line is that of an illustration: a subsidiary role in the service of the practices of the natural numbers. Still, it is a better-developed illustration that not only displays inductive infinity and discreteness, but operations as well. Therefore, the practice of interpreting an addition on the number line uses the number line not as an environment in which to work additive problems but as a display of a one-to-one correspondence between arithmetic sentences and number-line figures and, consequently, as a site where numerical platitudes such as commutativity and associativity may become problematic

Adding and subtracting zero

Tapia *et al* (1974) justify that $a + 0 = a$ in set-theoretic terms. Then, the text provides an "interpretation on the number line", where the zero addend is identified with a loop.

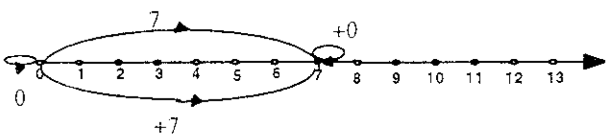

Figure 3
Addition of zero on the number line (p. 131)

In the case where the addition 0 + 7 = 7 is represented, the loop at 0 introduces a new object into the number-line register Previously, arrows were used to indicate a shift of position along the number line, and in the context of addition the difference between origin and extreme of an arrow was essential to portray the additive situation Within that context, a loop is not an arrow; as an ending loop (i e., in the representation of 7 + 0) it does not portray a counting situation, and as an initial loop (i e., in the representation of 0 + 7) it adds to the superfluity of the first arrow the fact that there is not even a different starting point for the counting action.

Still, no addition would be identified unless at least two arrows were drawn: therefore, the loops emerge out of the necessity to keep the one-to-one correspondence between arithmetic statement and number-line representation. The authors opt for strengthening the metaphor at the expense of making the number-line representation more complex: loops become an integral part of the number-line lexicon, providing a way to differentiate between the number-line counterparts of statements like 7, 0 + 7, 7 + 0, 0 + 7 + 0

Following Brousseau (1986), I would say the case of adding zero in Tapia *et al.* (1974) is an example of the effect of the *metadidactic* shift. [6] What had originally been brought in as a resource to understand the mathematics involved (the number line for the natural numbers) becomes an object of study in its own right and unfolds a structure of its own (zero is a loop, to add is to juxtapose two arrows, etc). Ernest (1985) observed a similar phenomenon when analyzing test items:

> It seems appropriate to say that the questions [. .] test children's knowledge of a particular model of addition, and test the ability to translate number sentences into representations using this model, and vice versa. (p. 418)

The shift is accentuated in the case of subtraction, which is described on the number line as translation to the left To represent subtraction, arrows in both directions are drawn in different colors (black for arrows pointing to the right and brown for arrows pointing to the left) *A priori*, the difference in colors does not seem to carry any extra meaning but rather to emphasize the difference in directions of the arrows. However, Exercise 35(c) asks the reader to "write the operations indicated on the number line" (p. 142) and presents Figure 4.

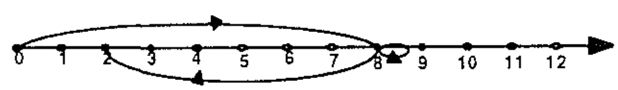

Figure 4
The subtraction 8 - 0 - 6 = 2 on the number line (p 142)

One can observe that in the event that loops and arrows were drawn in one color, two answers would be viable: 8 - 0 - 6 = 2 and 8 + 0 - 6 = 2. The way the exercise is presented (with arrows and loop in different colors) and the fact that the loop and the left-headed arrow are drawn in the same color seem to indicate that the colors are providing a new feature for the number-line arithmetic. Arrows are drawn in different colors in order to distinguish loops that represent +0 from loops that represent -0, thus keeping a one-to-one correspondence between arithmetic statement and number-line interpretation. In fact, the answer key provides only one correct answer for this exercise, "8 - 0 - 6 = 2" (p. 461). Hence, the existence of a distinct number-line representation for adding zero allows the property of the additive identity to emerge as more than a platitude about numbers: the *ad hoc* character of the adding-zero representation not only shows the existence of a correspondence but also collaborates in isolating an event to be called "the law of the identity element".

The one-to-one correspondence strengthens the metaphor
The preceding cases of number-line arithmetic illustrate how the text enriches the metaphor by achieving a one-to-one correspondence between statements of arithmetic operations and number-line interpretations. From those examples, one can conclude that the textual practices in which the number line is involved do not always clarify properties of the number system: they also, and notably, reaffirm the number line as a metaphor for the number system. This amounts, in particular, to producing number-line counterparts to all features of the number system, in much the sense as Black (1979) says that:

> [a] metaphorical statement [...] incites the hearer to select some of the secondary subject's properties; and [b] invites him to construct a parallel implication-complex that can fit the primary subject [...] (p. 29)

As a result, the increase in the number-line register is not a shift in content but a consistent push toward the structuralist goal: for example, the existence of null translations (that is, the inclusion of loops as if they were arrows) within the number-line register collaborates in reconceptualizing addition from being a process having to do with counting numbers to being an object whose properties have to be studied. In the following, I describe some specific knowledge claims in which the number-line metaphor is involved.

What does the number-line metaphor allow the text to say about numbers?

Black's (1979) *interaction view* describes metaphor as potentially responsible for the textual production (and, possibly, validation) of knowledge claims:

> The metaphorical utterance works by "projecting upon" the primary subject a set of "associated implications," comprised in the implicative complex, that are predicable of the secondary subject. [...] The secondary subject [...] determines a set of [...] current opinions shared by members of a certain speech-community. [... Also,] a metaphor producer may introduce a novel and nonplatitudinous "implication-complex." (pp. 28-29)

In the following, I describe several instances in which the line allows the text to make knowledge claims about the number system.

The number line as a site for justification of the rule of signs for multiplication
Tapia *et al.* (1974) construct multiplication of natural numbers from the Cartesian product of sets in Chapter 8. In Chapter 13, multiplication is defined for integers as pairs of natural numbers and interpreted in terms of the local metaphor of integers as natural numbers with a sign. Within the number-line interpretation for additive problems, the text says:

> Remember that now you have to consider two kinds of signs: the sign of the number and the sign of the operation. The sign of the number is interpreted thus:
>
> +5 [means] to count 5 units to the right
> -5 [means] to count 5 units to the left
>
> The sign of the operation is interpreted thus:
> + preserves direction, - reverses direction (p. 338)

Now, for the interpretation of multiplication, those rules are applied thus:

> $a \cdot (+2)$ [means] 2 times a in the same direction
> $a \cdot (-2)$ [means] 2 times a in the opposite direction

Observe how the following products are interpreted on the number line:

> (a) (+3)(+2) 2 times (+3) in the same direction
> (b) (-3)(+2) 2 times (-3) in the same direction
> (c) (+3)(-2) 2 times (+3) in the opposite direction
> (d) (-3) (-2) 2 times (-3) in the opposite direction
> (pp. 341-2)

The new features brought in by the representation of the integers on the number line (sign for the operation vs. sign for the number, change of direction vs. preservation of direction) combine with the features for number-line multiplication by means of textual symmetry. Every sign for the number must be combined with every possibility regarding direction. Although sentences (b) and (c) yield the same result, a term-by-term number-line interpretation of the arithmetic sentence (in terms of the unit being translated, the magnitude of the translation, and the direction of the translation) shows them to be different (actually, symmetric). The one-to-one correspondence between arithmetic statement and number-line interpretation is apparent here again, in agreement with the rest of the structural discourse about numbers (which promotes the status of numerical trivialities such as commutativity and associativity to that of *properties*).

The one-to-one correspondence between arithmetic statements and number-line interpretations, however, pays off by building a model that integrates the four kinds of multiplication sentences. The quoted descriptions of number-line multiplication provide number-line illustrations for the rules of signs stated previously in Tapia *et al.* (1974, p. 341). In the case of sentence (d), for example, the first factor indicates that the new unit is -3; the second factor says to translate that unit 2 times, but as this factor is negative, the translation is to be done in the direction opposite to that of the unit. Therefore, on the one hand, the number-line interpretation illustrates the multiplication of integers, but on the other hand, it provides a step-by-step algorithm for verification of the arithmetic result. From the reader's perspective, working within the number-line metaphor, the symmetric combination of movements on the number-line provides a tool for the validation of the rules of signs for multiplication. [7]

The number line as a site for the construction of the density of the rational numbers

Rational numbers are defined as equivalence classes of ordered pairs of integers and represented by the local metaphor of 'fraction of integers'. Under the heading 'Density of the Rationals', Tapia *et al.* (1974) recall the discreteness of the integers and then ask:

> How many rational numbers are there between 3/5 and 4/5? If we consider the fractions with denominator 5, there are none. But we can get fractions between them if we consider equivalent fractions with a larger denominator

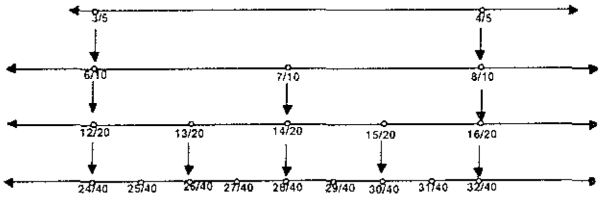

Figure 5
Density on the number line (p 405)

Finding a number between two given numbers

A first question to ask is: why do these two numbers (3/5 and 4/5) present a problem? Clearly, the textual assumption that they do invokes the notion of fraction as counting a new unit (fifths, in this case): between 3 and 4 there is no other number, and so "If we consider the fractions with denominator 5, there are none" (p. 405). But then equivalent fractions are recalled, and it is found that 3/5 = 6/10 and 4/5 = 8/10. The notion of fraction as counting with a new unit and the integer number line allow the authors to show 7/10 between 6/10 and 8/10 without any further discussion.

The issue, however, has not been addressed within the official discourse of the rational numbers. Although the definition of the *less than* relation had been previously stated (on p 403), it is not brought in either to find a fraction between the two given or to justify that the proposed fraction is between the two given. The formulation of the betweenness problem in terms of the number-line metaphor and the number-line fact that there must be a 7 between 6 and 8 allow the solution to emerge and be self-evident without any reference to the official discourse about order of rationals.

Figure 5 shows that the procedure of finding equivalent fractions with larger denominators is reiterated to produce more fractions between the two given. In all iterations, the number line is brought in to solve the betweenness problem and to allow the solutions to proceed without requiring a justification: the number line displays the fractions between the two given, and no question of their actual betweenness is posed

The whole figure (e g., the four number lines together in Figure 5) is also intended to relate the above procedure to the original problem. The numbers 24/40 and 32/40 are to be seen as fractions with denominator 40 in order to produce the solution: just as 27 is between 24 and 32, so 27/40 is between the two fractions. But they also need to be seen as equivalent to 3/5 and 4/5 in order to show that the obtained solution solves the problem that had been presented. Hence, the four-number-line figure is not just a juxtaposition of four number-line representations: it unites into a single whole two crucial pieces of knowledge (equivalence of fractions and the betweenness of integers) and produces a new object (betweenness of fractions) for the discourse on numbers without any explicit discussion of the consistency between order among the rational numbers and order among the integers

The existence of infinitely many rational numbers between two given rational numbers

The second notion addressed by Tapia *et al* (1974) is that of infinitude.

> As this process of multiplying by 2 [hence, of finding equivalent fractions] can be done indefinitely, it follows that between two rational numbers there are infinitely many rational numbers We state this property by saying that the set of rational numbers is dense (p. 405)

In other words, the rational infinitude (called 'density') is constructed in terms of the natural infinitude (induction) by the indication of iterating the process described in the previous paragraph For each iteration, the number line acts as before. The number line is a solving and justifying tool, legitimized for the authors by their knowledge of the real number system and legitimized for the reader by his or her knowledge of the integers' number line

The role of the Euclidean line in the construction of the number line for the rational numbers is crucial in understanding how density is constructed After incorporating the notion of fraction into the discourse on rational numbers, Tapia *et al.* (1974, p. 390) had recalled the integer number line and subdivided the unit. The fact that the subdivision of the given unit had gone without saying shows that the line is taken as a continuous whole to which the practices associated with the notion of fraction can be applied. Hence, the use of the notion of fraction and the intuitive completeness of the Euclidean line allow the reader to accept the construction of the number line for the rational numbers

The same numbering practices already used to produce the number line for the integers are used to produce the number line for the rational numbers: in the ostensive discourse, fractions appear as counting with a different unit. In fact, the figures accompanying the emergence of the number-line representation represent one fraction per number line, thus emphasizing the practice of counting with a new unit rather than the co-ordination of all possible units. Hence, the number line for the rational numbers emerges out of the integer number line, and the Euclidean line allows the new hash marks indicating numbers (discussed in the final section of this article) to be added without comment.

In this sense, one can see how the line is taken as more than a visual representation: the Euclidean line actually provides its own density features as background whence new numbers can be brought to the fore. However, it should also be noted that the iterative process that officially justifies density takes the line only as a background assumption (i.e., the point is there). The foreground argument deals with integers only: there is no explicit discursive indication that every point on the line would correspond to a rational number, because there is no situation where a point would be picked out and brought to the foreground without a prior justification related to the iterative process

The statement that there is no fraction (with denominator 5) between 3/5 and 4/5 and the way in which the betweenness problem is solved show how the text controls the density of the Euclidean line (that is, how it smoothly paces the emergence of identifiable rational numbers). An attempt by the authors to provide an actual number-line representation for all the rational numbers would eliminate the distinction between hash marks on the number line and points on the Euclidean line, creating a subsequent obstacle to the emergence of the real numbers. Confronted by that possibility, the number-line figures show only a few rational numbers at a time, thereafter preserving the notion of rational number as counting with a new unit

Therefore, the density of the rationals is produced by the number-line metaphor: the process of construction of the integer number line and the new unit defined by a particular fraction find in the Euclidean line a counterpart for any fraction. The solution to the betweenness problem described in the previous paragraphs allows such an infinite collection of fractions to display density

The integer number line and the progress toward constructing a bijection with the real numbers

The two consecutive texts that I have analyzed achieve a partial end for the discourse on numbers in Tapia *et al.* (1975), where the set of real numbers is defined as the set of all decimals and the bijection between the real numbers and the line is stated [8] The construction of the bijection as a mathematical notion requires leveling the status of numbers of different kinds (i.e., naturals, integers, rationals, decimals, irrationals) All of them are eventually displayed as being included in the set of real numbers, and hence, as being equivalent in the sense of belonging to the same set. The correspondence between that set and the line also has to be constructed Still, the text only says:

> To every real number corresponds a point on the line. To every point on the line corresponds a real number. There is a bijective function that maps the set of real numbers onto the set of points of the line (Tapia *et al.*, 1975, p 164)

The text provides the reader with a number-line figure where some few integers, fractions, decimals, and irrationals are placed. The practices by which the text had defined functions included primarily the giving of a rule of association between elements of different sets, or a formula, or a complete extension – be it by enumerating ordered pairs or by displaying a pair of Venn diagrams connected by arrows None of these is present when the correspondence between the line and the real numbers is stated; only the figure of the line and some points labelled on it illustrate the correspondence.

What allows that ostensive construction to go without saying? The number-line metaphor, introduced as one among many didactic devices to illustrate the discourse about numbers, becomes gradually accountable for the integration of the number sets and for the potential construction of the map that is eventually stated. The following paragraphs attempt to illustrate this claim

The number line for the integers

A first thing to notice here is that the number line is not accountable for the textual construction of the integers and does not appear as a natural graphical environment where such a construction should be restated. The textual construction of the integers consists of the definition of an equivalence relation in the Cartesian product N x N that is canonically represented as a lattice in Tapia *et al.* (1974, p. 330)

The emergence of the number-line representation for the integers is a product of the interaction between the natural number line and the identification of the integers as natural numbers with a sign. The authors say:

> We have represented the set of natural numbers on a ray with origin *o*, which corresponds to zero. (p 330)

Hence, the text recalls the emergence of the natural number line as a ray, although its emergence would officially mention (and present) it as a line. The replacement of *line* by *ray* amounts to incorporating into the explicit discourse an implicit feature of the construction of the natural number line that had been a current practice in the ostensive discourse (as the number-line figures for addition and subtraction of natural numbers had already used a ray, but would keep calling it a line).

The official natural-number-line representation on one ray allows the text to invoke the opposite ray. Therefore, the text can say:

> The integers are represented similarly. Given a line and a point *o* that corresponds to zero, the positive integers are represented on one of the rays and the negative integers on the opposite ray. Opposite numbers are symmetrically placed with respect to *o* on the number line That is why they are also called symmetric numbers. -3 and + 3 [....] are symmetric numbers.

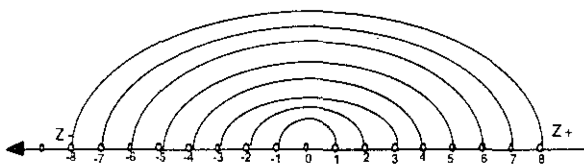

Figure 6

The number line for the integers (p. 329)

The emergence of a number line for the integers is based on the symmetric interplay of two enlargements and a correspondence: the natural numbers are enlarged to be natural numbers with a sign, the natural numbers are represented on a ray, the ray can be extended to the left; hence, the integers can be represented on the whole line.

The identification of the natural number line with a ray allows for an identification of the natural numbers with the positive integers (within the metaphor of the integers as natural numbers with a sign). In the explicit discourse, the natural number line is used to structure the integer number line on the basis of resemblance: as the set of integers contains two copies of the set of natural numbers (the positive and the negative), and the natural numbers would be represented on one of the rays with origin *o*, then those two copies of the set of natural numbers would be naturally placed on each ray with origin *o*

In the ostensive discourse, the natural number line also produces an identification between the natural numbers and the positive integers by means of an interplay between the picture of the natural numbers on a ray and Figure 6 (placed one right after the other on the same page). The natural number line (ray) and the ray representing the positive integers in Figure 6 have the same direction and are drawn in the same color (black), whereas the ray representing the negative integers in Figure 6 has the opposite direction and is drawn in a different color (brown).

Finally, the resemblance between the natural numbers and each of the two classes of natural-numbers-with-a-sign allows the natural number ray to create the two rays that produce the integer number line By pairing each positive integer with the negative integer that is based on the same natural number, Figure 6 brings to the fore a symmetric disposition of the number line. This symmetric disposition produces it as a whole with the same numbering unit without addressing (either explicitly or implicitly) the law of additive inverses. Therefore, the notion of symmetry invoked by the quotation to call the opposite numbers *symmetric numbers* alludes not to the fact that $+a + (-a) = 0$, but to the fact that they are represented on different rays with origin *o* and equidistant from *o* on the number line.

One has, therefore, a payoff of the number-line metaphor: its capacity to enlarge itself without losing its status of being a line, and hence to provide an environment where the hierarchical construction of the number system can still equate notions of number (such as that of natural and integer) which were incomparable in the official discourse Thus, the number-line metaphor allows the bijection between the line and the real numbers to be constructed in steps, in spite of the structural enlargements of the notion of number.

The correspondence question

After stating the density property of the rational numbers, Tapia *et al.* (1974) say:

> Think: you have already seen that to each rational number there corresponds a point on the number line and that between two rational numbers there are infinitely many rational numbers. Is it true that to every point on the line there corresponds a rational number? (p. 406)

That *correspondence question* has, in fact, emerged several times along the enlargements of the number systems. Just before introducing the number-line representation, Tapia *et al.* (1974, p. 114) stated as axioms of order on the line that there are two natural orders and that there are infinitely many points in between two given points Those statements provide them with a reason to remark, right after the introduction of the number-line representation, that the natural numbers are in one-to-one correspondence with a proper subset of the points of the line (p. 115)

The statement of the axioms of order promotes the notion of density in the Euclidean line to the status of a mathematical notion It also allows the authors to observe that the natural number line is properly included in the line. However, the status of that observation is only meaningful from the authors' perspective: the facts that the real numbers would eventually become a mathematical object equivalent to the line and that the real numbers properly contain the natural numbers legitimize the emergence of the correspondence question as a step toward the building of such a correspondence.

Hence, the emergence of the observation of the proper inclusion while describing the number-line representation for the natural numbers is foreign if one were dealing with only an illustration To exemplify it with another illustration: when a set is represented by a Venn diagram (that in a geometric context could also be recognized as a simple closed curve) and its elements represented as dots in the interior of the curve, the text does not ask what the other (non-distinguished) points in the interior of the curve represent.

The authors' position with respect to what they want to develop as knowledge of numbers provides a hint that can give us a better perspective: the line is one among many pedagogical devices that illustrate numbers (as the text also uses Venn diagrams and lattices occasionally), and it is also (but later) a distinguished mathematical object to use in stating a correspondence with the whole number system The eventual goal of establishing that correspondence justifies the authors' addressing the correspondence question; but from the reader's perspective, the line is still not accountable for those issues that will appear in the future. Hence, the emergence of the correspondence question is an outcome of the metaphorical interaction between the numbers and the line: an interaction that includes the detection of similarities and differences between systems that are basically distinct

The initial quotation for this sub-section shows that the distinction between the line as a representation and the line as the target of the correspondence is not ludicrous. Borges (1964, p. 90) tells the story of an empire where cartography had achieved such perfection that "the map of one Province alone took up the whole of a City"; but the cartographers were not satisfied, so they "set up a Map of the Empire which had the size of the Empire itself and coincided with it point by point". However, "Succeeding Generations understood that this Widespread Map was Useless" and abandoned it. Borges finishes his story by saying that apart from the ruins of that map "in the whole Country there are no other relics of the Disciplines of Geography"

In the case of the number line, the line as a representation had notably relied on the differentiation of two kinds of points, those used and those not used, by means of hash marks However, there is a crucial difference between the number-line representation of the integers and that of the rational numbers The procedure to represent the integers allows the reader to see all the number line from just a few numbers represented, as there is only one unit that is being translated in both directions.

On the other hand, the procedure for creating the number line for the rational numbers can be interpreted as creating, for each rational number, a new number line using the integer number line in two ways: the integer number line provides a unit segment that is assumed to be divisible into any number of parts and also acts as a metaphor to number the new hash marks originated by the partition in terms of the new unit. This process of numbering the new hash marks creates a new identification of some fractions (with a fixed denominator) with some points on the number line. The whole set of rational numbers, however, is not, either actually or potentially, represented on the number line, in order that the text does not lose control of the difference between hash marks and points on the line. The emergence of the number line for the rational numbers would be, in Borges' terms, too big a map for anybody to read The emergence of the correspondence question is a consequence of the need to keep the distance between the number system currently in use (the rationals) and the line

Yet from another standpoint, for each number system, the construction of the number-line representation and the explicit or implicit posing of the correspondence question are opposed with respect to the metaphorical relation between the number system and the line. In each case, the construction of the number line requires one to choose an origin and a unit These are arbitrary decisions legitimized as such by the fact that the line is a particular representation of the number system. The decisions reveal each number-line figure as one among many possible number-line representations

In contrast, the posing of the correspondence question assumes that the points used in the representation are identified with the numbers they represent In the case of integers, for example, the identification is explicitly stated: "The integers do not fill the line" (p. 329) This identification promotes the status of the line from being one among many possible representations to being the actual number system. The identification of the points with the numbers, rather than the representation of the numbers by the points, is what allows the correspondence question to emerge.

It is under that interaction that the detection of the differences between the integer number line and the Euclidean line (e.g., "the integers do not fill the line"), the statement of the density of the rational numbers, and the use of the same Euclidean line to represent both number sets, allow the possibility that the correspondence question could have an answer. Hence, the Euclidean line plays a crucial role in keeping the connection between:

(a) the overall knowledge of numbers and their operations; *and*
(b) the particular study of some number system.

When the correspondence question is stated for the rational number line, the line becomes the horizon and the tiller of all the enlargements. The number line, as a transactional object between past and future along the text's time-line, evolves from being one among many didactic devices to a mathematical object accountable for knowledge claims

The incompleteness of the set of rational numbers and the completeness of the real number system

The correspondence question is posed in Tapia *et al.* (1974), but its answer is deferred to the following course. In Tapia *et al.* (1975), the identification of the rational numbers as (infinite) repeating decimals allows the text to request the emergence of infinite non-repeating decimals and identify them as numbers. This request exhibits the action of a textual symmetry that produces the irrationals (the new, the different) under the assumptions that the existing numbers need to be completed within their environment (infinite strings of numbers with a decimal points) and that the completing elements are the same (i.e., numbers) although different [9]

The creation of irrational numbers by virtue of the use of the decimal representation of rational numbers allows the text to present a possible answer to the correspondence question:

> There is a property of the rational numbers that suggests we should answer this question 'yes': between two rational numbers there is always another rational number (pp. 163-4)

And immediately after, to declare it wrong:

> But then the irrationals appeared, and so we realize that the set of rational numbers has 'holes', that is, that there are points on the line that do not correspond to any rational number. (p. 164)

Therefore, the emergence of the irrationals as numbers is used in the text as a necessary condition for them to be placed on the number line. Because of the number-line metaphor, the identification of points on the line with numbers allows the text to establish the rational number line as incomplete. As all numbers are to be represented on the line, and distinct numbers are to correspond to distinct points, the emergence of irrationals as distinct from rational numbers implies that the points they occupy on the line are not occupied by rational numbers; therefore, the rationals do not fill the line.

The existence of irrational numbers seems to leave unquestioned that they should be placed on the number line. Two related discursive reasons allow that assumption to be unquestioned. The first reason is based on the discursive strategy used in the construction of the irrational numbers: the need for symmetry among decimal numbers. The fact that rational numbers have a number-line representation and the fact that terminating and repeating decimal numbers are just another way of writing the rational numbers imply the existence of a number-line representation for decimal numbers. The definition of the set of real numbers as the set of all decimals privileges the representational symmetry over the contexts of emergence of the different kinds of decimal numbers Within the frame of the set of real numbers providing a symmetric and complete whole, another symmetry establishes the placement of the irrational numbers on the line as unquestionable. Because the rational numbers are placed on the line, the irrational numbers, being just additional decimal numbers, should also be placed on the line.

The second reason is based on the recurrence of the line as a representation for the number sets. Tapia *et al* (1974, 1975) have constructed successive number-line representations by means of local metaphors (integers as naturals with a sign, rational numbers as fractions of integers, terminating decimal numbers as integers with a decimal point) that are no longer metaphors: when the rational numbers are constructed, the integers are already natural numbers with a sign, and the set-theoretic notion is not the principal subject any more. The successive reifications of those local metaphors reinterpret the statement that each number set contains an isomorphic copy of the number system that it extends: actually, each number set contains the number set that it extends.

As noted in the previous section, the number-line figures mirror those changes Although for each number set an *ad hoc* number line is constructed, when each local metaphor dies, the difference between the number line for a number set and its enlargement vanishes. The various number-line figures become one and the same, with different levels of graphical detail This unity of the number line is used thus: because the irrational numbers have been produced as numbers, they must lie on the same number line as well.

Therefore, as the line cannot show the 'holes' by itself, the authors create them by bringing the irrationals into the discourse about the line. As the irrationals are something different from what has been already represented on the line (rationals as decimals), the authors must let the holes exist so that the irrationals can fill them. However, as they are also numbers (and this similarity with the other numbers has become more important than the differences), it goes without saying that they should be represented on the line: the authors' 'creation of the holes' appears as the reader's 'discovery of the holes'.

Finally, the placement of the real numbers on the line as a consequence of the definitions of irrational and real numbers is a second consequence of the same symmetry: rationals and irrationals are seen as different-but-the-same, and rationals have been granted a number-line representation. Just as for the integers, the opposite ray to the natural number line had to be highlighted to be used by the new points; in the real number case, the holes in the rational number line are to be discovered as filled by the irrational numbers. The symmetry between rationals and irrationals backs up such a manipulation of the rational number line. Such a symmetry also pushes the metaphor toward its reification: nothing else can be conceived within the decimal representation of numbers, therefore, the text can say that the real numbers fill the line; moreover, the text can identify the real numbers with the line.

The fundamental theorem of similarity

The fundamental theorem of similarity (or Thales' theorem) is stated in Tapia *et al.* (1980) as "Any two segments on a line R are proportional to their parallel projections on another line R′" (p. 79). The theorem is proved by observing that parallel projection keeps equality, addition, and order of segments invariant, which allows the text to say that the subsets of segments of R and R′ are directly proportional magnitudes. However, their definition of scalar magnitudes makes use of the notion of real number for defining a unit to measure the length of every segment of R.

There is a qualitative difference between the proof given by Tapia *et al.* (1980) and those given in other texts for the same level. For example, Repetto, Linskens and Fesquet (1968) begin by saying:

> One considers an arbitrary segment *x*, but such that it is contained an exact number of times in OP [a segment on one of the lines] and another exact number of times in PQ [another segment on the same line] [...] (p. 32)

That is to say, Repetto *et al* (1968) assume commensurability That assumption is not uncommon in high-school mathematics textbooks. Alcántara, Lomazzi and Mina (1975), for example, start by assuming commensurability and after finishing the proof, observe:

> If AB and BC do not have a common submultiple, their ratio is an irrational number. In that case, the proof [...] requires the application of some concepts that are beyond the scope of this course (p 27)

By comparing these two other texts against Tapia *et al.* (1980), one can see the different roles that the notion of number plays in each. Both Repetto *et al.* (1968) and Alcántara *et al* (1975) are working inside a geometric context where the numbers correspond to actual geometric operations that they invoke as shorthand In that case, finding a common unit for two segments on a line means finding an actual segment that can be juxtaposed a finite number of times to cover both segments completely.

Tapia *et al.* (1980), however, do not refer back to numbers at all when proving the theorem. It is their assumption of measurement that once and for all puts any line into correspondence with the real numbers. The theorem becomes one of relating two lines by relating two correspondences between each line and the real numbers

The discourse of geometry in Tapia *et al.* (1974, 1975, 1980), although incorporating features from transformation geometry, had been following the Euclidean text very closely, thus delaying the incorporation of measurement practices The metaphoric interaction between the line and numbers, reversed in this case to be used in the discourse of geometry, allows the authors to convert the geometrical problem into a measuring problem where, in particular, the commonplaces about numbers (addition, order, and so forth) and the correspondence with the line are used to ensure that measures exist even though units cannot be counted. The resort to the one-to-one correspondence in this case has an important consequence: it achieves a proof of a true proposition that includes the incommensurable case without dealing explicitly with it

Conclusion

This article has shown instances of the work of the number-line metaphor in the discourse of a textbook. That work can be summarized by saying that the number-line metaphor is the horizon and the tiller of all the enlargements of the notion of number. As such, the emergence of correct mathematical notions and the validation of true mathematical statements become accountable for in terms of it. As noted in the last section, once reified as a mathematical notion, it allows the textbook authors to supply a proof of Thales' theorem that neither assumes commensurability nor makes the incommensurable case explicit.

However, this study only took the uses of this particular metaphor in these particular texts as examples. It used those examples to show a way to produce an internal critique of mathematical textbooks; that is, a critique that targets the practices in which the notions are involved within the text. It did so by considering the textbook as a system of discursive practices, which is one more metaphor among the many that may apply.

The study also showed the mathematical discourse of a textbook as an environment where one can find a mathematical discourse that is subject to a regime possibly different from the official discourse of mathematicians The knowledge claims, which the number-line metaphor accounted for their emergence or justification, were brought in as different instances where true conclusions are arrived by means that are textually correct but whose logic does not follow the steps of the usually accepted logic of contemporary mathematical discourse. To be quite clear, the point has not been to show any defect in the work of Tapia *et al* (1974, 1975, 1980, 1989), for whom I confess great admiration. The point has been rather to exhibit how even in a text that so faithfully attempts to follow what is assumed to be mathematical knowledge, one can find a distinct (and consistent) regime of mathematical discourse that differs from the one in the academic community of reference.

Notes

[1] This article is based on parts of my M A thesis (Herbst, 1995) carried out under the direction of Dr Jeremy Kilpatrick A previous version was presented at Working Group 10, ICME 8, Seville, Spain Thanks to Jeremy Kilpatrick, David Wheeler, and Geoffrey Howson for valuable comments on previous drafts.

[2] Kang (1990) used the theory of didactic transposition (Chevallard, 1991) in studying textbooks (see also Kang and Kilpatrick, 1992)

[3] Until the educational reform of 1995, high school in Argentina consisted of five years equivalent to U S Grades 8-12 Each year required a mathematics course that was a prerequisite for the following year's course.

[4] Tapia *et al.* (1980) and Tapia *et al* (1989) respectively

[5] All quotations from Tapia *et al.* (1974, 1975, 1980) have been translated from Castillian by me.

[6] My translation of the French term *glissement metadidactique*

[7] The previous introduction of the rule of signs on p 340 had shown that minus times minus is plus from the ordered pairs version of the integers. A direct translation from ordered pairs to natural numbers with a sign had allowed the text to state the rule of signs in the usual arithmetic form. No formal or informal justification via the extension of functions and the preservation of properties had been invoked so as to justify defining multiplication in that particular way.

[8] In Tapia *et al.* (1989), the authors revise all previous constructions and provide a construction of the real numbers by nested intervals However, the partial end achieved in Tapia *et al* (1975) has some consequences in the later use of the notion of real number in Tapia *et al.* (1980)

[9] The quoted construction of irrationals may be indisputable from a mathematician's perspective; my point is that its correctness is a historical and discursive achievement To illustrate it, consider the comment from Stifel about the same subject quoted in Kline (1980, p 114). (The original piece dates from 1544.)

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