Open-Ended Tasks as a Trigger for Mathematics Teachers' Professional Development

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The vision expressed in the NCTM's *Curriculum and evaluation standards* [NCTM, 1989] and *Professional teaching standards* [NCTM, 1991] encourages most mathematics teachers to teach mathematics differently than the ways in which they were taught as students [Leinward, 1994]. It is suggested that teachers' conceptions of teaching are strongly influenced by the way in which they themselves have learned the subject matter [Kagan, 1992; Stofflett & Stoddart, 1994]. Thus, it is particularly important to create for them learning situations in which they engage in powerful learning of mathematics, followed by reflection on the processes that take place. In this spirit, Clarke [1994] discusses the need to "model desired classroom approaches during in-service sessions to project a clearer vision of the proposed changes" [p. 38, ibid.].

In designing such learning situations for mathematics teachers we face two conflicting considerations regarding the choice of content to be used: On the one hand, selecting an advanced mathematical topic which is new to the teachers can be powerful in terms of their own learning experience, however, they may remain with the feeling that such experience is only possible with high level mathematics that appears richer than most parts of the secondary curriculum. In such cases we often hear teachers claim that implementing such ideas could only be done on special occasions or for special students, not as an integral part of the ordinary curriculum that must be "covered". On the other hand, selecting familiar topics from the secondary mathematics curriculum may not create real learning experiences for mathematics teachers who already know and even teach these topics.

In this paper we suggest a way to bridge between these two conflicting arguments. We propose a tested approach to creating rich and powerful learning situations for teachers (as well as for students) by modifying standard tasks based on familiar mathematical content from the secondary curriculum and turning them into open-ended ones with multiple correct answers. Socha [1991], inspired by Judah Schwartz, refers in a general manner to the kind of approach presented and analyzed here. In addition to the role that such problems play in teacher development processes, the proposed approach to generating such open-ended problems can easily be used by teachers for implementation in their own classrooms, because it is quite straightforward and can be applied to many standard tasks available in common textbooks.

Evidence about the power of such modified tasks was collected in in-service workshops for secondary mathematics teachers within the framework of a professional development program aimed at introducing innovative approaches to teaching mathematics. In this paper one task is deeply analyzed to illustrate the potential of this approach. Many other tasks of the same nature were used with similar results (see a sample of tasks in the Appendix). The analyzed task deals with *quadratic and linear functions*—a part of the standard curriculum which teachers are both familiar with and often teach.

The task

A standard task	A modified task
How many intersection points does the parabola: $y = x^2 + 4x + 5$ have with the straight line: $y = 2x + 5$?	Find an equation of a straight line that has two intersection points with the parabola: $y = x^2 + 4x + 5$

Solution strategies of the standard task

The standard task is fairly common in the middle school mathematics curriculum. Clearly, graphical technologies, which are becoming available to more and more mathematics classrooms, enable students to obtain the answer to the standard task by observing the graphs of the two given functions. However, even if a graphical representation is displayed students are usually requested to augment their observation by a rather procedural analytical solution derived by solving a system of two equations. After all, the two intersection points may not show at all, or may be so close to each other that they seem as if they coincide.

This standard task is a problem with only one correct answer that does not lend itself to much classroom discussion and discourse, while the modified task is a slightly altered version that turns it into one with multiple correct answers and prompts mathematical communications as well as many other desirable learning situations in the spirit of the NCTM Standards [1989, 1991].

Solutions strategies of the modified task

We turn to a description of the solution processes that took place at one of the workshops in which the (above) modified task was presented. This particular task was used in a number of in-service workshops with different groups of mathematics teachers, and always facilitated powerful learning processes and interactions. Although formally the modified task only calls for one specific answer, i.e., an equation of one line, the participants were not ready to move on to another task before thoroughly exploring this particular task. They were anxious to search for much more than one answer, pose more questions stemming

from the task, try various strategies, and find more families of straight lines satisfying the task. Several approaches were suggested by the participants, each approach led to new questions, investigations and generalizations. The order in which the various approaches are presented reflects the order that occurred at the workshop.

Five main solution strategies were suggested in order to find an equation of a straight line intersecting the given parabola at two distinct points:

Strategy No. 1:
Choose any point above the vertex of the parabola, then find the equation of the line parallel to the x-axis that passes through the selected point (see Figure 1)
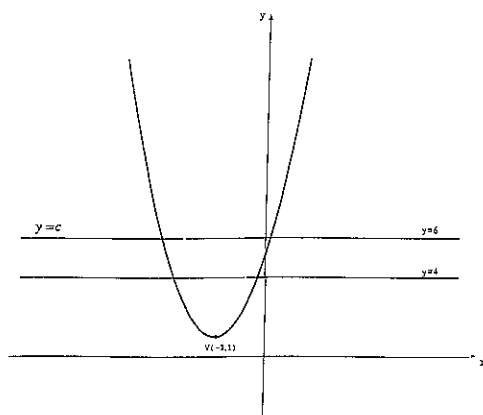

Figure 1

This strategy led to a general (infinite) family of lines: $y = c$ where $c > 1$, that satisfy the requirement of the task, i.e., each one has two intersection points with the given parabola.

Two questions were posed:

1.1 What happens when $c = 1$?

1.2 How can we obtain the equations of other straight lines that do not belong to this infinite family and that intersect the given parabola at two different points?

Strategy No. 2:
Choose any (non-zero) number for the slope m, then find the equation of the linear function with the chosen slope m passing through the vertex of the parabola (see Figure 2).

This strategy led to the following questions:

2.1 Will any number we choose for a slope lead to an appropriate line? What if we choose zero for the slope? What if we choose a very large ("steep") slope?

2.2 How can the family of lines obtained in such a way be represented in a general form?

2.3 Can the approach suggested by this strategy work if we choose a different point A instead of the vertex? In such a case, can any number m serve as a slope for an appropriate line?
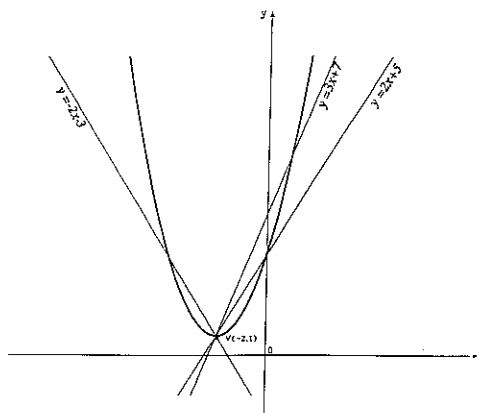

Figure 2

The last part of question 2.1 stimulated much discourse and conflict, because several teachers thought (erroneously!) that for a very large slope the line would not intersect the parabola at another point. This common misconception [Leinhardt, Zaslavsky, & Stein, 1990; Zaslavsky, in press] was revealed and treated by the group members on their own It was an opportunity for many of the teachers to refine their understanding of possible connections between a parabola and a straight line and to experience the need to come up with strong arguments to convince themselves and each other.

Question 2.2 led to a general (infinite) family of lines: $y = m(x + 2) + 1$ *where* $m \neq 0$, that satisfy the requirement of the task An interesting observation was made indicating that this family has no common element (i e, a straight line) with the family obtained by Strategy 1.

As to question 2.3—it led to a generalization of the condition for the slope m: Instead of the requirement that $m \neq 0$ for lines through the vertex, the condition was extended to *m not equal to the slope of the tangent through A,* i.e., not equal to the derivative.

Strategy No 3:
Choose any line obtained by Strategy 2, translate it upward by t, then find its equation (see Figure 3)
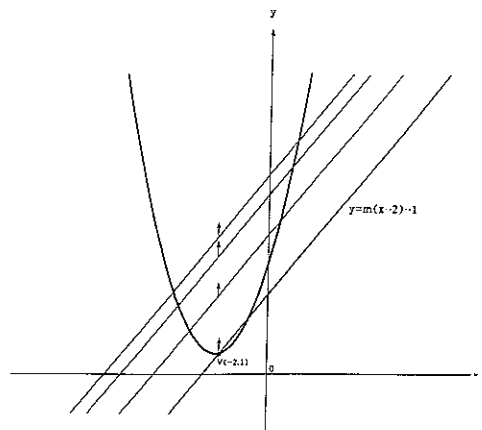

Figure 3

This strategy led to more questions:

3.1 How can the family of lines obtained in such a way be represented in a general form?

Question 3.1 led to a general (infinite) family of lines: $y = [m(x + 2) + 1] + t$ *where* $m \neq 0$, $t \geq 0$, that satisfy the requirement of the task

Stemming from the last conclusion the next question was posed:

3.2 Does the last family: $y = [m(x + 2) + 1] + t$ *where* $m \neq 0$, $t \geq 0$, include the equations of *all* the lines intersecting the given parabola at two different points?

Question 3.2 led to the analysis that for every point A on the parabola there are an infinite number of straight lines that intersect the parabola at two points and do not belong to the family: $y = [m(x + 2) + 1] + t$ *where* $m \neq 0$, $t \geq 0$. Because if m' is the derivative at point A, then all the lines with slope m' which are between the tangent through A and the line $y = m'(x + 2) + 1$ (parallel to the tangent through the vertex) do not belong to the above family (see Figure 4)
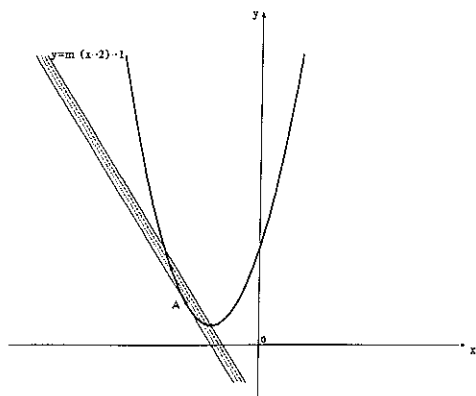

Figure 4

Strategy No. 4:
Choose any point A "inside" the parabola, then find the equation of any linear function passing through the selection point A (see Figure 5)
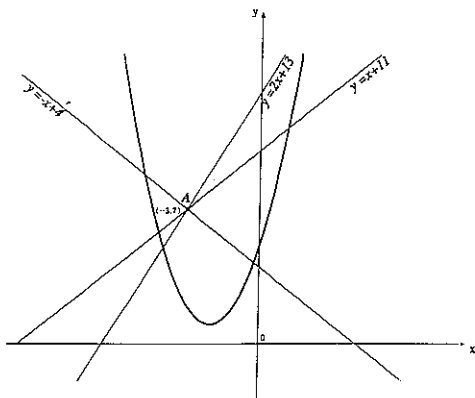

Figure 5

Once again, several questions were posed with reference to this strategy:

4.1 Do all linear functions passing through A intersect the parabola at two different points?

As with question 2.1, this question prompted a lot of debate among the group Judging by what appeared graphically, many teachers thought (erroneously!) that there are straight lines through the point A that are not parallel to the y-axis and that have only one intersection point with the parabola (the one that appears in Figure 6)
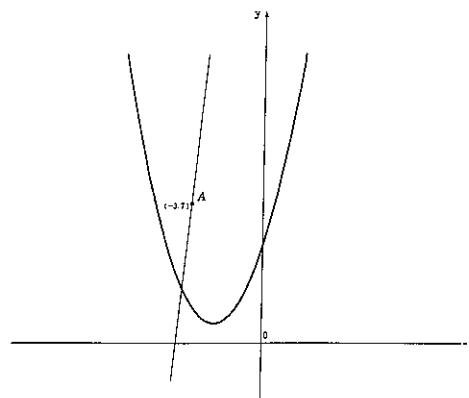

Figure 6

The discussion led to another question:

4.2 What if the given parabola were "wider", say, $y = 1/2x^2 + 2x + 3$? Does the graph of every linear function passing through a point A "inside" this "wider" parabola intersect it at two different points?

Again, this was a tough question, since the graph of the linear function (see Figure 7) seemed as if it would never "reach" the parabola, that is, it would not intersect the parabola at a second point
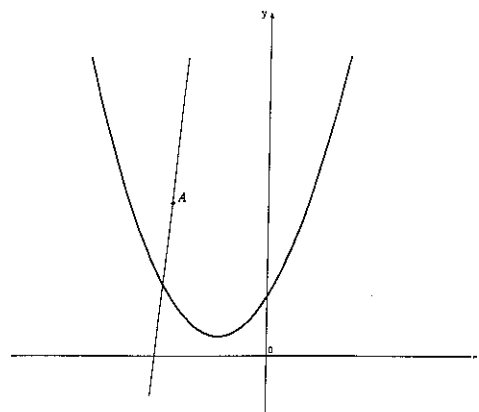

Figure 7

Strategy No. 5:
Choose any two points on the parabola, then find the equation of the straight line through both of them (see Figure 8).
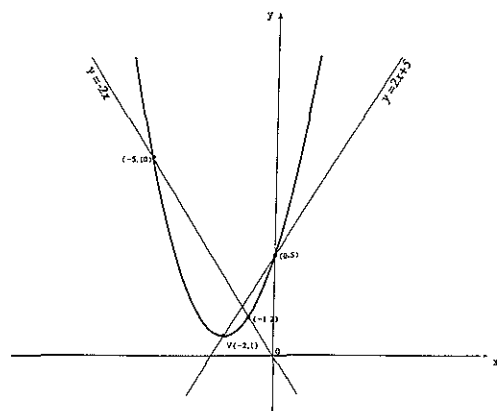

Figure 8

This strategy led to the following questions:

5.1 Which two points should we choose in order to get a line belonging to the family that we obtained by Strategy 1: $y = c$ *where* $c > 1$?

5.2 Can we obtain by this Strategy the equations of all the lines intersecting the parabola at two different points?

5.3 How can we represent the whole class of equations of lines intersecting the parabola at two different points?

The answer to question 5.3 was not trivial and required a high level of generalization and a lot of algebraic manipulations. By group efforts the conclusion was reached that the entire family of linear functions intersecting the parabola at two distinct points could be defined as: (1) Every line the equation of which can be represented as $y = (a + b + 4)x + 5 - ab$, *where* a and b are real numbers, and $a \neq b$.

From this followed another question:

5.4 What if the condition $a \neq b$ is replaced by the condition $a = b$?

The answer to question 5.4 led to the identification of the family of all tangents to the parabola, i.e., the set of all linear functions intersecting the parabola at exactly one point. This family was defined as: (2) Every line the equation of which can be represented as $y = (2a + 4)x + 5 - a^2$, *where* a is a real number. Consequently, it was suggested that each line included in family (1) could be generated by translating upward a tangent of family (2) (see Figure 9). Since the given parabola is concave up an equivalent way to define family (1) would be: (3) Every line the equation of which can be represented as $y = (2a + 4)x + 5 - a^2 + d$, *where* a is a real number, and $d > 0$.
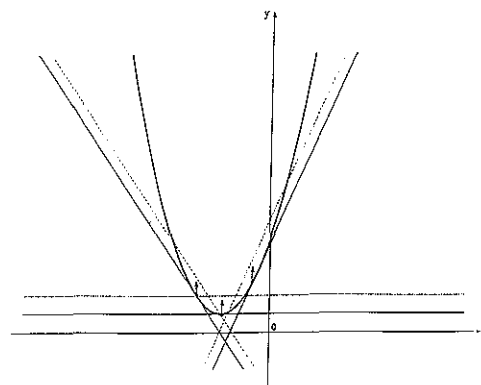

Figure 9

A final remark by a participant of the group referred to the fact that all this broad investigation was done for just one specific parabola. It was obvious that it could have been done for any other given parabola, thus, the findings could be generalized at even a higher level. No one felt the need to treat the more general case. It seemed like a generic example that one could see through the underlying principles and ideas of the general case.

Reflection

The use of reflection as means of promoting teacher professional growth and development has been widely advocated [Robinson, 1989; Schön, 1987; Cooney, 1994; NCTM, 1991]. Following this view, reflection was incorporated as an integral part of each of the in-service workshops. The reflection that took place right after the full completion of the particular task analyzed above supports the notion that such tasks provide a rich context for teacher development. Both the teachers and the instructor reflected on the processes that they had encountered, first spontaneously, then in a more structured way. Everyone shared the feeling that this was both exciting and challenging. They felt that this was a true collaborative work, although they were not assigned to work in groups. Many expressed their wish that their own students would undergo similar processes.

The more structured part of the reflection called for analysis of the following: (i) The task, in terms of its nature and the kind of mathematics on which it called; (ii) The teachers as learners, in terms of individual and group learning processes and interactions including their motivation to learn; (iii) The instructor as a teacher, in terms of her role in creating the learning situation that took place; (iv) The teachers as teachers, in terms of possible implementation of similar situations and ideas in their own classrooms.

The task

Analysis of the task focused on the richness that a slight change in a standard task created. The kind of modification that was done for this particular standard task, i.e., elimi-

nating a piece of information that is usually provided, seemed applicable for many other standard tasks (as Socha [1991], claims). The fact that the task had multiple correct answers encouraged teachers to compare answers, to check the validity of their answers, and to search for relationships between different answers. The deep and broad mathematics concepts that were involved in solving the modified task suggested a way to create connections across a wide scope of mathematical ideas (e.g., families of functions, parameters, graphs, equations, slope, derivative, tangent, transformation, proof) at different levels of generalization and abstraction.

Teachers as learners

Analysis of the learning processes that the teachers encountered focused on individual differences along several dimensions such as: Approaches to the task; Contributions to the mathematical discourse; Views of what constitutes a convincing argument; Difficulties and misconceptions. These processes increased the teachers' awareness of individual differences and the role and legitimacy of errors in the learning process. Another direction of reflection dealt with the strength of the group as a whole and the nature of the collaborative work that took place and empowered each member of the group. A shared feeling was that the group effort led to an accomplishment that probably would not have been achieved by any one individual alone. Special attention was given to the activity of problem posing that occurred spontaneously throughout the workshop in the spirit of Brown & Walter [1993], and to its potential in stimulating discussion and further investigations as well as in providing opportunities for "personalizing mathematics" [Silver, 1994] Another point that was raised was the deep involvement in the task that everyone shared. In a way it was surprising for them to realize that they had learnt so many new things about a very familiar topic that they often teach. The notion that they, as teachers, will always have more to learn, even with respect to what they think they already know, gave new meaning to what "knowing" entails.

Instructor as teacher

Analysis of the role of the instructor in the workshop focused on her ways of facilitating mathematical discourse. The group attributed the learning situation that developed to the choice and design of a worthwhile open ended task. They also were convinced that the fact that the instructor did not interfere much and was not trying to direct the discussion according to a certain plan contributed to the richness and diversity that evolved. The instructor actually modeled a way to foster communication and social interactions, and did not transmit a sense of authority. On the contrary—they noticed that at several points the instructor was faced with questions she had not anticipated and was not sure what the "right answer" would be. she often took part in the discussion as an ordinary member of the group. The instructor confessed that she had not dealt with questions such as 3.2, 5.3, and that she felt comfortable enough to tackle them together with the rest of the group.

Teachers as teachers

the entire group of teachers kept moving from reflection on themselves as learners to what it would take for them to create similar learning situations for their own students. the fact that the task they engaged in was part of the mathematics curriculum meant that they could actually use this particular one and try it out in their classrooms (most of them did, and reported on how this went later on at other meetings). They tried to take several other standard tasks from mathematics textbooks, and came up with a large collection of open ended tasks which seemed worthwhile and that could stimulate mathematical discourse. They articulated what it was that the instructor modeled for them and how their conception of what a teacher should or could do was affected. Another point that was raised dealt with the nature of explaining and convincing. Their own experience seemed to sharpen their need to be more attentive to individual cognitive structures. They reflected on the instances in which one of them was trying to convince the other with what appeared to him as a convincing argument while, in fact, it did not convince his colleague. In this particular task the differences were mostly routed in the dominant representation with which they reasoned more willingly (graphically versus algebraically). The nature of the task, having multiple correct answers at different levels of generalization, also seemed to be a way to handle students' differences by allowing each one to answer at the level that interested him or her most.

Conclusion

This paper carefully examines one typical in service workshop for mathematics teachers which was part of a sequence of similar workshops aiming at modeling ways to implement the NCTM Standards [1989, 1991]. The design of powerful learning situations for the teachers, the kind that they rarely if ever encountered as students, sets the grounds for reflection that fosters new conceptions of the role of the mathematics teacher and of the potential of open-ended mathematical tasks. The particular method of designing such mathematical tasks, which is plausible and within reach, encouraged teachers to implement similar tasks in their own classrooms on a regular basis. They began sharing their experiences with their colleagues and constructing a collection of such tasks that work in the classroom. Through such activities teachers seemed to develop both aspirations to change and the confidence that they can teach differently, as Cooney [1994, p. 19] suggests, "so that they can envision that the bridge to a different classroom is close and crossable"

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Appendix
A slight modification that makes a big difference

A standard task	A modified task
Factor $x^2 - 3x - 4$	Find a last term that makes $x^2 - 3x - ?$ factorable
Cancel the following fraction $\dfrac{x - 5x - 6}{x^2 - 3x - 4}$	Find two last terms that make $\dfrac{x^2 - 5x - ?}{x^2 - 3x - ?}$ cancelable
Find the intersection point between the graphs of $y = 3x - 8$ and $y = -2x + 7$	Find an equation of a straight line through the point (3,1)
Find the x-intercepts of the parabola $y = x^2 - 7x + 10$	Find an equation of a parabola passing through the points (2, 0) and (3,-2)
Find the vertex of the parabola $y = 3x^2 - 12x + 11$	Find an equation of a parabola the vertex of which is (2,-1)
Sketch the graph of the function $y = x^3 + x$	Give an example of a real function the derivative of which is positive for any real number
Find the inverse function of $f(x) = 2x + 1$	Give an example of a function that is equal to its inverse

[W]e should not be so concerned with motivating everyone to do well in mathematics but, rather, with giving everyone a chance to find out whether he or she is interested in doing mathematics To reject the study of mathematics as a free and well-formed decision is the choice of a responsible citizen; to plod through it docilely is a slavelike response and to drop out without reflective consideration is to lose an opportunity both to learn mathematics and to learn about oneself. In a politicized classroom, students become citizens who have some control over their academic lives This means promoting dialogue both within mathematics lessons and about mathematics as a potential avenue of self-affirmation.

Nel Noddings