Supporting the Participation of English Language Learners in Mathematical Discussions

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The aim of this paper is to explore how teachers can support English language learners in learning mathematics and not only English The focus of the analysis will be on one important aspect of learning mathematics: participation in mathematical discussions. I use a discourse perspective (Gee, 1996) on learning mathematics to address two pairs of questions central to mathematics instruction for students who are learning English (as well as for students who are native English speakers):

- What can a teacher do to facilitate student participation in a mathematical discussion? How can a teacher support students in speaking mathematically?

- What are the variety of ways that students talk about mathematical objects? What are the different points of view students bring to mathematical situations?

I examine a lesson from a third grade mathematics discussion of the geometric shapes from a tangram puzzle to illustrate how one teacher supported mathematical discussion and describe how students participated by talking about mathematical situations in different ways I use excerpts from the transcript of this lesson to exemplify supportive teaching strategies and describe the variety of ways that students communicate mathematically In particular, the teacher did not focus primarily on vocabulary development but instead on mathematical content and arguments, as he interpreted, clarified and rephrased what students were saying

During this lesson, the teacher's instructional strategies included: 1) using several expressions for the same concept; 2) using gestures and objects to clarify meaning; 3) accepting and building on student responses; 4) revoicing (O'Connor and Michaels, 1993) student statements using more technical terms; and 5) focusing not only on vocabulary development but also on mathematical content and argumentation practices My analysis of student participation in the discussion shows that students brought several different ways of talking about mathematical objects and points of view of mathematical situations to the classroom discussion. Two important functions of productive classroom discussions are uncovering the mathematical content in student contributions and bringing different ways of talking and points of view into contact (Ballenger, 1997; Warren and Rosebery, 1996).

Frameworks for examining mathematical discussions

One view of learning mathematics is that English language learners need to focus primarily on learning how to solve word problems, understand individual vocabulary terms and translate from English to mathematical symbols (e.g. Mestre, 1988; Spanos, Rhodes, Dale, and Crandall, 1988) This view is reflected in many current recommendations for mathematics instruction for English language learners that emphasize vocabulary and comprehension skills (Olivares, 1996; Rubenstein, 1996; MacGregor and Moore, 1992). These recommendations provide a limited view of learning mathematics and do not address a current increased emphasis on mathematical communication

In contrast, in many mathematics classrooms, students are no longer primarily grappling with acquiring technical vocabulary, developing comprehension skills to read and understand mathematics textbooks or solving standard word problems. Instead, students are now expected to participate in both verbal and written practices, such as explaining solution processes, describing conjectures, proving conclusions and presenting arguments

I use a discourse perspective on what it means to learn mathematics to consider the participation of English language learners in mathematical discussions, and take a view of discourse as more than sequential speech or writing, using Gee's definition of 'Discourses' [1]:

> Discourses are ways of being in the world, or forms of life which integrate words, acts, values, beliefs, attitudes, social identities, as well as gestures, glances, body positions and clothes. (Gee, 1996, p. 127)

Participating in classroom mathematical discourse practices can be understood in general as talking and acting in the ways that mathematically competent people talk and act when talking about mathematics in a classroom, and involves much more than the use of technical language. Gee uses the example of a biker bar to illustrate this In order to look and act like one belongs in a biker bar, one has to learn much more than a vocabulary While knowing the names of motorcycle parts or models may be helpful, it is clearly not enough to participate in a biker bar community In the same way, knowing a list of technical mathematical terms is not sufficient for participating in mathematical discourse.

Beyond technical terms, mathematical discourse includes constructions used to prove or explain statements such as "If x, then y", "Let x be the case", "Assume . ", "This is the

case because . ", "to make comparisons, such as 'the higher
..., the smaller ..'" and to describe spatial situations. There
are also valued discursive practices, such as abstracting and
generalizing, making precise statements and ensuring
certainty, and the need to acquire control over the accompa-
nying forms which reflect them (see Halliday, 1978 for some
instances)

One important aspect of classroom mathematical dis-
course practices is participating in mathematical
discussions, either as a whole class or with a group of stu-
dents. For the purposes of this analysis, a mathematical
discussion will be taken to be:

> purposeful talk on a mathematical subject in which
> there are genuine pupil contributions and interactions.
> (Pirie, 1991, p 143)

While during a classroom mathematical discussion the con-
versation might involve some aspects of standard or
canonical mathematical discourse, discussion in a classroom
is not the same as the discourse of research mathematicians.
Students in mathematics classrooms should not be expected
to talk and act in the same ways that research mathemati-
cians do Nevertheless, in general, students participate in
mathematical discussions or communicate mathematically
by making conjectures, presenting explanations, construct-
ing arguments, etc. about mathematical objects, with
mathematical content, and towards a mathematical point
(Brenner, 1994).

We can use several lenses to examine how English
language learners (or other students) participate in mathe-
matical discussions and how teachers support student
participation in these discussions. Below I summarize the
instructional strategies the US NCTM standards (1989) and
certain mathematics education classroom research (e g. Ball,
1991; Cobb *et al*, 1993; Silver and Smith, 1996) suggest
for orchestrating and supporting mathematical discussions:

- model desired participation and talk; support these
  when displayed by students;
- encourage student conjectures and explanations;
- call for explanations and evidence for students' state-
  ments;
- focus on the process not only the product;
- compare methods, solutions, explanations;
- engage students in arguments for or against a state-
  ment (move beyond "agree" or "disagree");
- encourage student-to-student talk;
- ask students to paraphrase each other's statements;
- structure activities so that students have to understand
  each other's methods.

A comparison of this discourse approach to mathematical
discussions with recommended language development
strategies for teaching mathematics to students who are
learning English yields some reassuring similarities and
some disturbing differences. Below is a brief summary of
some commonly recommended language development
strategies:

- use realia, pictures, models, diagrams, gestures,
  charts, labels, dramatization, etc;
- provide different forms of participation (orally, in
  writing, graphically) and interaction (with the teacher,
  with peers, alone);
- adjust speech (speed, enunciating, vocabulary);
- check for comprehension;
- provide feedback, appropriate response and adequate
  wait time;
- provide resources whenever possible and useful (dic-
  tionaries, translations, interpreters)

I find these language development strategies disturbing in
that there is little guidance on how to concentrate on
the mathematical content of discussions. In general, the
consideration of mathematical content for English language
learners is apparently limited to providing exposure to
content vocabulary and identifying differences in mathe-
matical conventions (such as the use of a comma or a period
for decimals or the different placement of the long division
sign).

Another drawback of some language development
approaches to mathematics teaching is a focus on correc-
tion of vocabulary or grammatical errors (Moschkovich, in
press), obscuring the mathematical content in what students
say and the variety of ways that students who are learning
English do, in fact, communicate mathematically Instead,
by focusing on mathematical discourse, teachers can move
beyond focusing on errors in English vocabulary or gram-
mar to hear and support the mathematical content of what
students are saying

Although the strategies summarized above can be helpful
in orchestrating discussions, they are, however, only a
beginning These guidelines have some serious limitations.
First, both the NCTM standards and the language devel-
opment strategies can be interpreted as presenting
manipulatives and pictures as 'extra-linguistic clues',
assuming that the meaning of an object can be communi-
cated without using language Instruction cannot depend on
using manipulatives as a way to deal with either compre-
hension or conceptual problems. Students need to use
language in context, participate in conversations and clarify
the meaning of the objects they are manipulating, looking
at or pointing to. Second, neither set of recommendations
provides enough guidance for how to listen to and under-
stand the different ways that students talk mathematically
or for how to bring together these different ways of talking
in classroom discussions.

The lesson excerpts presented below provide examples
of strategies for supporting a mathematical discussion
among English language learners and contrast the variety
of ways that students talk about mathematical situations
They come from a third grade bilingual classroom in an
urban California school. In this classroom, there are thirty-
three students who have been identified as Limited English
Proficiency. In general, this teacher introduces students to
concepts and terms in Spanish first and then later conducts
lessons in English. Students are surrounded by materials in

both Spanish and English and desks are arranged in tables of four so that students can work together.

The students have been working on a unit on two-dimensional geometric figures. For several weeks, instruction has included technical vocabulary such as 'radius', 'diameter', 'congruent', 'hypotenuse' and the names of different quadrilaterals in both Spanish and English. Students have been talking about shapes and have also been asked to point to, touch and identify different instances. This lesson was identified by the teacher as an ESL mathematics lesson, one where students would be using English in the context of folding and cutting exercises to form tangram pieces (see Figure 1). It also illustrates how, when the goal is supporting student participation in a mathematical discussion, listening to the nature and quality of students' mathematical discourse is as important as (if not more so than) focusing on students' English language proficiency.
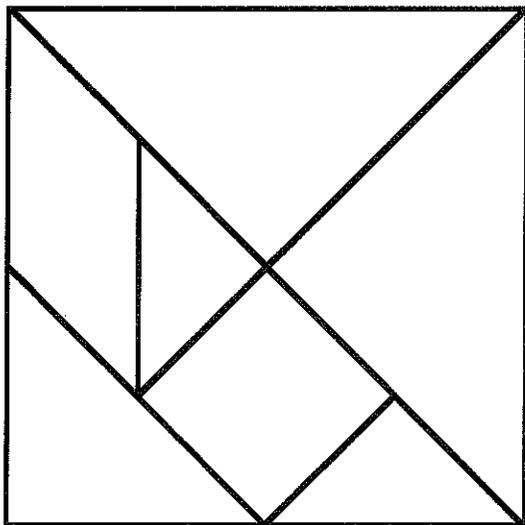

Figure 1   A tangram puzzle

The goal of my analysis of this lesson was two-fold. One was to identify teaching strategies during an actual classroom lesson in order to provide genuine illustrations, while the other was to identify examples of the different ways English language learners communicate mathematically and the different points of view they bring to a discussion, in order to provide a basis for hearing and understanding these students better during classroom discussions.

I present a partial transcript of this lesson in four excerpts. The first segment involves descriptions of a rectangle. The second concerns comparing a rectangle and a triangle, while the third involves comparisons of a parallelogram and a trapezoid in a small group and the fourth details a whole-class discussion of whether a trapezoid is or is not a parallelogram. I have numbered conversational turns for ease of reference, and [] indicates non-verbal information to help with deictic references, such as 'this' or 'that'. In excerpt 3, () indicates an English translation of student remarks in Spanish. T is the teacher and Ss students.

How is the teacher supporting participation in a mathematical discussion?

Excerpt 1: Describing a rectangle

1. Teacher: Today we are going to have a very special lesson in which you really gonna have to listen. You're going to put on your best, best listening ears because I'm only going to speak in English. Nothing else. Only English. Let's see how much we remembered from Monday. Hold up your rectangles... high as you can [students hold up rectangles] Good, now Who can describe a rectangle? Eric, can you describe it? [a rectangle] Can you tell me about it?

2. Eric: A rectangle has. two . short sides, and two .. long sides

3. T: Two short sides and two long sides Can somebody tell me something else about this rectangle? If somebody didn't know what it looked like, what, what . how would you say it?

4. Julian: Parallel(o) [holding up a rectangle]

5. T: It's parallel Very interesting word Parallel, wow! Pretty interesting word, isn't it? Parallel. Can you describe what that is?

6. Julian: Never get together. They never get together [runs his finger over the top length of the rectangle]

7. T: What never gets together?

8. Julian: The parallela ... the ... when they go, they go higher [runs two fingers parallel to each other first along the top and base of the rectangle and then continues along those lines] they never get together

9. Antonio: Yeah!

10. T: Very interesting. The rectangle then has sides that will never meet. Those sides will be parallel. Good work. Excellent work. Anybody else have a different idea that they could tell me about rectangles?

11. Student: Another different [unclear] parallelogram.

12. T: It's called a parallelogram, can you say that word?

13. Ss: Parallelogram.

14. T: What were you going to say, Betsy?

15. Betsy: Also a parallelogram it calls a rectangle.

16. T: A parallelogram is also a rectangle? They can be both?

17. Betsy: Yeah

18. T: Wow, very interesting. Can you convince me that they can be both?

19. Betsy: Because a rectangle has four sides and a parallelogram has four sides

20. T: [unclear]

21. Eric: [unclear] a parallelogram

22. T: You want to borrow one? [a tangram piece] I really want to remind you that you really have to listen while your classmate is talking ...

23 Eric: Because these sides [runs his fingers along the widths of the rectangle] will never meet even though they get bigger, and these sides [runs his fingers along the lengths of the rectangle] will never meet even though they get bigger And these sides [picks up a square] will never meet [runs his hand along two parallel sides] and these sides will never meet. [runs his hand along the other two parallel sides]

24. T: When you say get bigger you mean if we kept going with the line? [gestures to the right with his hand]

25 Eric: Yeah.

26 T: Very interesting

During this lesson, the teacher employed some important strategies to orchestrate and support students' mathematical talk. [2] In general, he established and maintained norms for discussions, asking students to listen to other students, to agree or disagree, to explain why they believed something and to convince the teacher of their statements. The teacher also used gestures and objects, such as the cardboard geometric shapes, to clarify what he meant For example, he pointed to vertices and sides when speaking about these parts of a figure.

Although using objects to clarify meanings is an important ESL instructional strategy, it is crucial to understand that these objects do not provide 'extra-linguistic clues' The objects and their meanings are not separate from language, but rather acquire meaning through being talked about and these meanings are negotiated through talk. Although the teacher and the students had the geometric figures in front of them, and it seemed helpful to use the objects and gestures for clarification, students still needed to sort out what 'parallelogram' and 'parallel' meant by using language and negotiating common meanings.

The teacher focused not only on vocabulary development but also on supporting students' participation in mathematical arguments by using three other instructional strategies that focus more specifically on mathematical discourse First, the teacher prompted the students for clarification: for example, in turn 16 the teacher asked a student to clarify the relationship between two geometric figures, and in turn 7 the teacher asked Julian to clarify what he meant by "they". Second, the teacher accepted and built on student responses, as can be seen in the above exchanges. In another

example (turns 4-5), the teacher accepted Julian's response and probed what he meant by "parallel".

Last, the teacher revoiced student statements, by interpreting and rephrasing what students said (turn 10 in relation to turn 8) Julian's "the parallela, they" becomes the teacher's "sides" and Julian's "they never get together" becomes "will never meet" The teacher moved past Julian's unclear utterance and use of the term "parallela" and attempted to uncover the mathematical content in what Julian had said He did not correct Julian's English, but instead asked questions to probe what the student meant..

Julian's utterance in turn 8 is difficult both to hear and interpret. He uttered the word 'parallela' in a halting manner, sounding unsure of the choice of word or of its pronunciation His voice trailed off, so it is difficult to tell whether he said 'parallelo' or 'parallela' His pronunciation could be interpreted as a mixture of English and Spanish; the "ll" sound being pronounced in English and the addition of the "o" or "a" being pronounced in Spanish. The grammatical structure is also intriguing The apparently singular "parallela" is followed by a "the ... " and then with a plural "when they go higher" [3] In any case, it seems clear that this was a difficult contribution for Julian to make and that he was struggling to communicate in a meaningful way

Excerpt 2: Comparing a rectangle and a triangle
Students were folding the rectangle and cutting it into a folded triangle and a small rectangle Holding these two pieces, the teacher asked the students to tell him how a triangle differs from a rectangle

56 T: Anybody else can tell me something about a rectangle that is different from a ... a triangle that's different from a rectangle? Okay. Julian?

57 Julian: The rectangle has para ... parallelogram [running his fingers along the lengths of the rectangle], and the triangle does not have parallelogram

58 T: He says that this [a triangle] is not a parallelogram How do we know this is not a parallelogram?

59. Julian: Because when this gets ... When they get, when they go straight, they get together [runs his fingers along the two sides of the triangle]

60. T: So, he's saying that if these two sides were to continue straight out [runs his fingers along the sides of the triangle], they would actually intersect, they would go through each other Very interesting. So, this is not a parallelogram and it is not a rectangle. OK.

During this short exchange, the teacher once again revoiced a student statement. In turn 58, the teacher restated Julian's claim that "the triangle does not have parallelogram" as "this is not a parallelogram", and in turn 60, the teacher restated Julian's claim that "when they go straight, they get together" as "if these two sides were to continue straight out, they would actually intersect, they would go through each other".

There are at least two ways that a teacher's revoicing can support student participation in a mathematical discussion The first is that it can facilitate student participation in gen-

eral, by accepting the student's response, using it to make an inference and allowing the student the right to evaluate the correctness of the teacher's interpretation of the student contribution. This move maintains a space open for further student contributions in a way that the standard classroom initiation-response-evaluation (IRE) pattern does not

The second way is that a revoicing move serves to make and keep the discussion mathematical. In revoicing, a student statement is often reformulated in terms that are closer to the standard discourse practices of the discipline. For example, in the first excerpt the teacher uses the term "sides", which is more specific than Julian's phrase "the parallela", because it refers to the sides of a quadrilateral, rather than any two parallel lines. This revoicing seems to have an impact on Julian who later uses the term "side(s)" twice when talking with another student (see excerpt 3, turn 79). In the second excerpt, the teacher changes Julian's colloquial phrase "get together" to "meet", which is more formal and therefore more aligned with standard school mathematical discourse. These two reformulations serve to maintain the mathematical nature of the discussion.

Revoicing is not, however, as simple as I have presented it so far. It is not always easy to understand what a student means. Sometimes, a teacher and a student speak from very different points of view about a mathematical situation. Even though in the second excerpt the teacher once again revoiced student statements, this time there seems to be more going on than just interpreting and re-stating what a student said. To understand this excerpt, it is important first to consider Julian's use of the phrases "has parallelogram" and "does not have parallelogram" and the teacher's responses to Julian. It is not clear from this excerpt whether Julian was using the expression "has parallelogram" to mean 'has parallel sides', as in "the rectangle has parallel sides" or to refer to the geometric figure "parallelogram", as in "the rectangle is a parallelogram" (i.e. has the parallelogram property). (During a later conversation with another student Julian (turn 85) seems to use the phrase "it's a parallelogram" to mean 'the lines are parallel'.)

At one point (turn 58), the teacher interpreted Julian's utterance "does not have parallelogram" as "is not a parallelogram". However, in turn 59, Julian seems to be referring to parallel lines. By turn 60, the teacher seems to be on the same track as Julian without having focused directly on the specific meaning of the phrase "does not have parallelogram" or having corrected Julian's turn 57 contribution.

One way to compare the teacher's and Julian's contributions during the second excerpt is to consider the ways of talking and points of view each brings to the discussion. Julian seemed to be telling a story about the lines, saying "when they go higher, they never get together" and enacting the story by running his fingers along the parallel lines. In contrast, the teacher was using the future tense, which sounds more predictive or hypothetical, saying for example, "*will* never meet" and "*will* be parallel".

The teacher and Julian were also talking about parallelograms from different points of view. While the teacher first referred to a category at turn 58, in response Julian narrated the situation once again, describing a property of the parallelogram (turn 59). The teacher then rephrased Julian's

response in a more hypothetical way using the subjunctive "if ... were .., they would ...". He also explicitly marked both revoicings, by using the tag "So, he's saying that ..." (in turn 60) and "He says that " (in turn 58)

While it is not completely clear what Julian originally meant in turn 57 or what the consequences of the revoicing in turn 58 were, it does seem that the teacher and Julian were bringing different points of view to the discussion. This difference in the points of view of the situation (which reappears in a later discussion about a trapezoid and a parallelogram in excerpt 4) may have contributed to the difficulty in interpreting and revoicing Julian's statement.

There is one more way that this teacher supported student participation. During the discussion in both excerpts 1 and 2, we see instantiated the teacher's general stance towards student contributions. The teacher moved past Julian's confusing uses of the term 'parallela' or the phrase "has parallelogram" to focus on the mathematical content of Julian's contribution. He did not correct Julian's English, but instead asked questions to probe what the student meant. This response is significant in that it represents a stance towards student contributions which can facilitate student participation in a mathematical discussion: listen to students and try to figure out what they are saying. When teaching English language learners, this may mean moving beyond vocabulary or grammatical errors to listen for the mathematical content in student contributions. (For a discussion of the tensions between these two, see Adler, 1998.)

What is the mathematical content of this discussion?

As students participated in this discussion, they were grappling not only with the meaning of individual words and phrases, but also with some important ideas about quadrilaterals and lines. It may be easier to see the mathematical content in this discussion by referring to the textbook's definitions for some of the concepts invoked during the discussion. The definition of parallel lines is "straight lines in a plane which have no common point no matter how far they are extended". The definition of a parallelogram is "a quadrilateral with two pairs of parallel sides".

One important idea that students seemed to be grappling with was class inclusion, for example when students described a square (which is also a rectangle, a parallelogram and a trapezoid), a rectangle (which is also a parallelogram and a trapezoid) or a parallelogram (which can also be a trapezoid, depending on which definition is used [4]). Students also seemed to be sorting out whether they were talking about a property of the figure, as in the phrase "it has parallel sides", or a category for the figure, as in the phrase "it is a parallelogram". These two points of view, one focusing on a property and the other on a category, may be especially important to sort out when talking about parallelograms, since the word "parallel", which describes a property of parallelograms, is part of the word "parallelogram", which describes a category of geometric figures.

Students were also grappling with the concept of parallel lines. One important aspect of parallelism is that one needs to imagine, hypothesize or predict what will happen to the

two lines segments if they are extended indefinitely.
Although two line segments may not meet here and now,
what matters when deciding if two lines are parallel or not
is whether the imagined lines would ever meet

Different ways of talking about a mathematical situation

Excerpt 3: Comparing a parallelogram and a trapezoid in small groups

> 78. T [to whole class]: What do we know about a trape-
> zoid. Is this a parallelogram or not? I want you to take a
> minute, and I want you at your tables, right at your
> tables. I want you to talk with each other and tell me
> when I call on you, tell me what your group decided.
> Is this a parallelogram or not?
>
> 79. Julian: [Julian and Andres have several shapes on
> their table: a rectangle, a trapezoid and a parallelogram]
> *Porque si. Nomás estas* (Because … Just these) sides
> get together [runs his fingers along the two non-parallel
> sides of the trapezoid, see Figure 2] *pero de este* (but on
> this side only) [runs his fingers along the base and top
> parallel sides of the trapezoid]
>
> 80 Mario: *Y este lado no* (And not this side)
>
> 81. Andres: *No porque mira, aqui tiene un lado chico*
> (No because, look, here it has a small side) [points to
> the two non-parallel sides of the trapezoid] *y un lado
> grande y tiene cuatro esquinas* (and a large side and it
> has four corners).
>
> 82. Julian: See? They get together, *pero acá no* (but
> not here) [runs his fingers along the base and top par-
> allel sides of the trapezoid]
>
> 83. Andres: *Acá no*. (Not here)
>
> 85. Julian: It's a parallelogram. Only this side, but this
> side meet [runs his fingers along the non-parallel sides
> of the trapezoid] [5]
>
> 86 T: [joins group] They would meet?
>
> 87 Julian: Yeah, but these sides, they won't. [runs his
> fingers along the base and top or parallel sides of the
> trapezoid]
>
> 88. T: OK. So one pair of sides meets, the other don't.
> So it is not, or it is?
>
> 89. Andres: No. It is.
>
> 90. T: It is?
>
> 91. Andres: Yeah.
>
> 92. T: Think about it now. I want you to talk about it
> Remember, a parallelogram from what you said was,
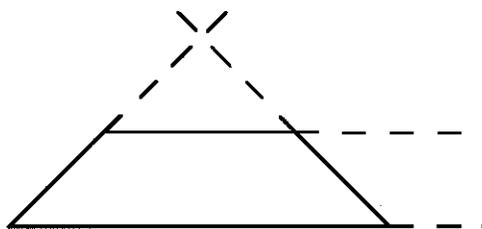> two sides, two pairs of sides

Figure 2   Julian describing a trapezoid (turns 79, 82, 85)

The students in this classroom brought a variety of ways
of talking about mathematical situations to this discussion.
In the three excerpts presented above, we have examples of
students talking about mathematical objects in narrative,
predictive and argumentative ways. For example, Julian (see
Figures 2 and 3) seemed to be telling a story about the par-
allel lines and using his fingers to enact the story, an
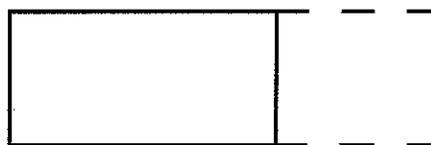example of a narrative approach

Figure 3   Julian describing a rectangle (turn 8)

Eric, in turn 23 (see Figure 4), was also telling a story. His
use of the future tense "will" and the phrase "even though"
makes this story sound more predictive and hypothetical.
His way of talking can be described as a combination of a
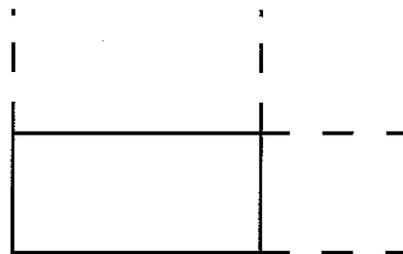narrative and a predictive approach

Figure 4   Eric describing a rectangle (turn 23)

In contrast, in turn 81, Andres employed certain linguis-
tic elements of the form of an argument, such as his use of
"because" and "look", but neither the mathematical content
nor the mathematical point of his argument were clear.

Narrative and predictive ways of talking are not presented as a dichotomy. Notice that Eric's contribution was a combination of these two ways of talking. Although it may seem easy to identify hypothetical ways of talking as mathematical, imagining and narrating can also be a part of standard mathematical discourse practices: for example, mathematical conversations about objects in four dimensions involve imagining and narrating a situation.

One way to think about the relationship between narrative and formal mathematical ways of talking is that a mathematical text can be embedded within a narrative. Solomon and O'Neill (1998) claim one of the characteristics of a mathematical text is that the mathematical argument is:

> atemporal [... and] achieves cohesion through logical rather than temporal order. (p. 216)

Seen in this manner, elements of both Julian's and Eric's contributions are similar to mathematical texts in that both contributions are atemporal statements that will always be true ("the lines will never get together" or "these sides will never meet") and are cohesive because of their logical rather than temporal structuring.

The point of the comparisons above is not to privilege one way of talking over another nor to decide which way of talking is more mathematical. In a classroom discussion, various ways of talking can contribute in their own way to the mathematical discussion and bring resources to the conversation. The point of these comparisons is to diversify our view of the different ways that students talk about mathematical objects and situations, to uncover the mathematical aspects of what students are saying and to be able to hear better the variety of ways in which students can communicate mathematically.

Different points of view of a mathematical situation

The final excerpt shows the different points of view that the teacher and the students brought to defining a parallelogram and a trapezoid. It also provides another example of how this teacher, rather than emphasizing language development, expected the students to focus on mathematical content.

Excerpt 4: Deciding whether a trapezoid is a parallelogram or not

> 92. T: [to the whole class] OK. Raise your hand. I want one of the groups to tell us what they do think. Is this a parallelogram or not, and tell us why. I'm going to take this group right here.
>
> 93. Vincent: These two sides will never meet, but these two will.
>
> 94. T: How many agree with that. So, is this a parallelogram or not?
>
> 95. Ss: Half.
>
> 96. T: OK. If it is half, it is, or it isn't?
>
> 97. Ss: Is.
>
> 98. T: Can we have a half of a parallelogram?
>
> 99. Ss: Yes.

> 100. T: Yes, but then, could we call it a parallelogram?
>
> 101. Ss: Yes.
>
> 102. T: What do you think? If we remember what Julian said. Would you repeat what you said at the beginning, a parallelogram is what?
>
> 103. Julian: It's a parallelogram's [meet] they never meet.
>
> 104. T: How many pairs of sides never meet?
>
> 105. Andres: Five.
>
> 106. Julian: Two. They have two sides . to . that . a square has . . [takes a rectangle] these sides [runs his fingers along the lengths of the rectangle], they not meet, and these sides [runs his fingers along the widths of the rectangle], never meet. That's a parallelogram.
>
> 107. T: Did everybody hear what he said?
>
> 108. Ss: Yeah.
>
> 109. T: There have to be two pairs of sides that never meet. Let's see if this is or not. Would these sides ever meet? [holding up the trapezoid]
>
> 110. Ss: Yeah.
>
> 111. T: Would they ever meet?
>
> 112. Ss: No.
>
> 113. T: No. Would these sides if we extend them up, would they ever meet?
>
> 114. Ss: Yeah.
>
> 115. T: OK. That's what you were saying. So, is this a parallelogram?
>
> 116. Ss: No.

During this excerpt, the students and the teacher brought together two points of view concerning defining a parallelogram. They differ in terms of what types of definitions are acceptable. While for the students "half a parallelogram" was an acceptable specification, this was not acceptable to the teacher. Rather than reflecting an error in students' reasoning or their lack of English proficiency, this exchange uncovers two fundamentally different views of this mathematical situation.

The standard definition of a trapezoid is "a quadrilateral with one pair of parallel sides" [6] and the standard definition of a parallelogram is "a quadrilateral with two pairs of parallel sides". The teacher's initial question "Is this a parallelogram or not, and tell us why?" assumed that this is an either/or situation. The teacher's point of view was dichotomous: a given figure either is or is not a parallelogram.

In contrast, some students seemed to have a dynamic point of view of this situation. Their response to the question "Is this a parallelogram or not?" was "Half", implying that a trapezoid is half of a parallelogram. This is a reasonable definition if it is understood to mean that since a parallelogram has two pairs of parallel sides, and a trapezoid has one pair of parallel sides, then a trapezoid is a half of a parallelogram because a trapezoid has half as many parallel sides as a parallelogram (see Figure 5). The students' point of view is also different in that they are focusing on whether and how these two figures possess the *property* of having pairs of parallel lines, rather than on whether the figures belong to one of two *categories*: "figures with two pairs of parallel lines" or "figures with one or no pair of parallel lines"
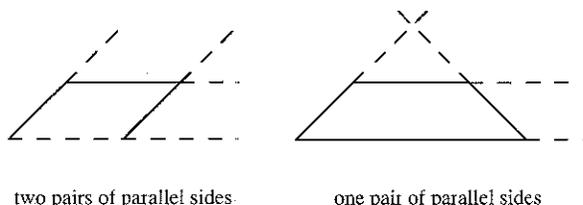

two pairs of parallel sides   one pair of parallel sides

Figure 5   A trapezoid is half a parallelogram

This interpretation of "a half parallelogram" would be consistent with Julian's earlier usage of the phrase "is a parallelogram" to mean "has parallel sides" If "is a parallelogram" means "has parallel sides", then a trapezoid has one half as many pairs of parallel sides as a parallelogram

Another way to see the students' point of view as dynamic is to consider a trapezoid as a parallelogram in transformation. We can take a trapezoid and transform it into a parallelogram
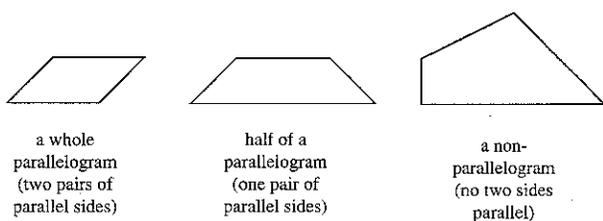

| a whole parallelogram (two pairs of parallel sides) | half of a parallelogram (one pair of parallel sides) | a non-parallelogram (no two sides parallel) |

Figure 6   A dynamic point of view

There are many possibilities for where this lesson might go next in a way that honors both mathematical discourse and the mathematical content of student contributions. One possibility is to encourage more student-to-student talk, asking students to consider and address each other's contributions. Another possibility is for the teacher to continue building on student contributions while using ways of talking which are closer to standard mathematical discourse,

as this teacher did when saying that for a figure to be a parallelogram "there have to be two pairs of sides that never meet"

Another strategy might be to talk explicitly about different ways of talking, asking students to consider different ways of describing parallel lines and defining a trapezoid by contrasting the teacher's contributions and the descriptions proposed by students. The teacher could explicitly compare descriptions focusing on the properties of a figure with descriptions focusing on the categories a figure belongs to, contrasting saying "a figure has parallel sides" with saying "a figure is a parallelogram". The discussion could also focus on explicitly comparing descriptions of a trapezoid as "being a half a parallelogram" with "having half as many parallel sides as a parallelogram", thereby working on grammatical constructions from within a content-focused discussion

A third way to talk explicitly about talk is to compare student ways of talking with textbook definitions, helping the student to see their descriptions in relation to the more regimented discourse of mathematics texts. During such a discussion, the teacher could explicitly compare student descriptions such as "the lines never get together" or "a trapezoid is half a parallelogram" with textbook definitions of parallel lines, parallelograms and trapezoids

Conclusions
The excerpts presented above illustrate several instructional strategies that can be useful in supporting student participation in mathematical discussions, such as establishing and modeling consistent norms for discussions, revoicing student contributions, building on what students say and probing what students mean. The teacher did not focus primarily on vocabulary development but instead on mathematical content and arguments as he interpreted, clarified and rephrased what students were saying.

What does it mean to say that mathematical discourse is more than vocabulary and technical terms? A discourse approach to learning mathematics means considering the different ways of talking about mathematical objects and points of view of mathematical situations that students bring to classroom discussions. There are a variety of ways in which students argue, provide evidence or present an argument. Sometimes they predict, imagine or hypothesize what will happen to an object. Sometimes they focus on categories of objects and other times on the properties of these objects. Students may have different points of view of what an acceptable description or definition of a mathematical object is.

A discourse approach to learning mathematics can also help to shift the focus of mathematics instruction for English language learners from language development to mathematical content. The lesson presented here shows that English language learners can and do participate in discussions where they grapple with important mathematical content. It is certainly difficult to consider carefully the mathematical content of student contributions in the moment. However, it is possible to take time after a discussion to reflect on the mathematical content of student contributions and design subsequent lessons to address this

content. But, it is only possible to uncover the mathematical content in what students say if students have the opportunity to participate in a discussion and if this discussion is a mathematical one.

This teacher provided this opportunity by moving past student grammatical or vocabulary errors, listening to students and trying to understand what students said. He kept the discussion mathematical by focusing on the mathematical content of student contributions, asking students for clarification, accepting and building on student responses and revoicing student statements.

Revoicing can be difficult to carry out, perhaps especially when working with students who are learning English. It may not be easy or even possible to sort out what aspects of a student's utterance are a result of the student's conceptual understanding or of a student's English language proficiency. However, the analysis of this lesson suggests that, if the goal is to support student participation in a mathematical discussion, determining the origin of an error is not as important as listening to the students and uncovering the mathematical content in what they are saying.

Acknowledgments

Notes
[1] While Gee distinguishes between 'discourse' and 'Discourse', I will use lower case 'discourse' here to refer to what Gee labels upper case 'Discourse'

[2] Although there are many valued instructional strategies focusing on either language development or mathematical content that the teacher did not use in this lesson (such as keeping track of student conjectures and conclusions on the blackboard or supporting more talk between students), the goal of this analysis is not to focus on what the teacher did not do, but rather on how he did, in fact, support participation in a mathematical discussion.

[3] There is a difference between English and Spanish in terms of how "parallel' is used as an adjective. While in English "parallel" can usually only be singular (one line is parallel to another; two lines are parallel), in Spanish, "paralelo/a" can be singular or plural (estas dos lineas son paralelas; una linea es paralela a otra) This utterance of Julian's may be further complicated by the fact that sometimes Spanish speakers do not pronounce a final "s" sound, so that it may be difficult to tell whether this utterance was in fact singular or plural. While it may be interesting to consider these differences, it seems unreasonable to jump to any simple conclusions about Julian's difficulties with this word

[4] I have run into two definitions of a trapezoid One is "a quadrilateral with exactly one pair of parallel sides", while the other is "a quadrilateral with at least one pair of parallel sides".

[5] As mentioned earlier, in this line Julian seems to be using "parallelogram" to mean 'parallel'

[6] The definitions of trapezium and trapezoid (a quadrilateral with no sides parallel) are often interchanged. In Spanish, ". the word trapezoid is reserved for a quadrilateral without any parallel sides, whereas trapezium is used when there is one pair of parallel sides. (This is opposite to American English usage)" (Hirigoyen, 1997, p. 167)

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