PRODUCTIVE AMBIGUITY IN THE LEARNING OF MATHEMATICS

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"'Ambiguity' is in itself an ambiguous word." – Leonard Bernstein

A formative experience of my teenage years was watching a worn-out film that my music teacher was eager for us to see of the conductor and composer, Leonard Bernstein, talking passionately about the role of ambiguity in music (Bernstein, 1976) I had never previously considered ambiguity as anything more than an annoyance and a failure of communication As chiefly a science and mathematics student, I admired plain speaking and wondered why poets, for instance, could not simply state clearly and concisely what it was they were trying to say. At times, I suspected that those who loved to speak in vague and colourful ways were simply disguising with purple prose their lack of anything much to say I was drawn to the precision and economy of mathematics yet, at the same time, I also loved music Bernstein's series of talks persuaded me that I had completely failed to appreciate an enormously important aspect of the arts (Tormey & Tormey, 1983)

An ambiguity is different from mere vagueness or woolly imprecision. Byers (2007) has offered the following definition of ambiguity, adapting one originally offered by Arthur Koestler for creativity:

> Ambiguity involves a single situation or idea that is perceived in two self-consistent but mutually incompatible frames of reference. (p. 28)

In the arts, an ambiguity may set up a lively tension between parallel ideas or provoke a crisp resolution; Bernstein frequently referred to ambiguities as beautiful. Musical ambiguities can relate to key (polytonality) or timing (polymeter). A good example of musical ambiguity is the tritone (half an octave), which is a restless-sounding interval, seeking some kind of resolution It can expand to a minor sixth or contract to a major third. Figure 1 shows an example of both possibilities, in the one case implying C major and in the other, F# major. [1] Heard alone, the tritone could lead to either key and musical suspense is created by not knowing which way it is going to go. Within a particular musical context, one resolution might seem more likely, giving the composer the option of either fulfilling our expectations or provoking surprise

Figure 1. Musical ambiguity: two different resolutions of a tritone

While ambiguity may be delightful in the arts, it seems common sense to many that ambiguities are not at all desirable in a discipline such as mathematics. The stereotypical mathematician is straight talking, matter-of-fact and to-the-point and therefore unlikely to appreciate delicate ambiguities. It is doubtful that any mathematics paper has been praised for its ambiguity; on the contrary, an elegant proof is more often seen as one that is efficient, precise and direct (Dreyfus & Eisenberg, 1986). Gupta (2001) describes the conventional contrast perceived between the arts and the sciences:

> Mathematical metaphors are powerful analytical tools precisely because of the unequivocal relationships between their components, whereas the power of the literary metaphor derives from the incertitude in the connections between its parts. (p. 589)

More recently, however, an important role for ambiguity has been proposed in the sciences and mathematics. Byers (2007) has argued persuasively that at one level, there is no difference between the arts and the sciences – both are creative human endeavours:

> Ambiguity plays a role in mathematics that is analogous to the role it plays in art – it imbues mathematics with depth and power. Ambiguity is intrinsically connected to creativity (p. 11)

Indeed, Byers has gone much further and argued for the *necessity* of ambiguity in mathematics:

> Ambiguity is not only present in mathematics, it is essential. Ambiguity, which implies the existence of multiple, conflicting frames of reference, is the environment that gives rise to new mathematical ideas. (p 23)

This is a strong statement, but it is amplified and supported to a considerable degree by Grosholz (2007) in her comprehensive summary and perceptive analysis of instances of what she terms "productive ambiguity" in the course of the history of science and mathematics:

> When distinct representations are juxtaposed and superimposed, the result is often a productive ambiguity that expresses and generates new knowledge. (p 25)

An ambiguity is not the same as an error, a paradox, a contradiction, an absurdity or a fallacy. An ambiguity derives from a significant degree of uncertainty, caused by a lack of specification regarding a particular feature or an unstated assumption, paradigm or frame of reference. This results in an ability to see the same situation in more than one way

There are many kinds of mathematical ambiguity:

1 *Symbolic ambiguity,* such as the same letter being used to stand for two different things or the × sign being used for multiplication of real numbers or for the cross product of vectors. This sort of ambiguity is typified by jokes such as:

$$\frac{\text{si}\cancel{n}\,x}{\cancel{n}} = six = 6$$

I include in this category ambiguities resulting from notational abbreviation, such as missing subscripts on partial derivatives to say which variables are held constant (*e.g.*, $\frac{\partial Q}{\partial T}$ in thermodynamics, which could be $\left(\frac{\partial Q}{\partial T}\right)$ or $\left(\frac{\partial Q}{\partial T}\right)$) or omitted limits on summation signs (*e.g.*, Σx), which may present no difficulties if only one meaning is taken, or may fatally interfere with understanding.

2 *Multiple-solution ambiguity,* such as with inequalities (*e.g.*, $x > 7$, but *how much* greater?) or the two roots of a quadratic equation. Multiple roots, for example, can be viewed either as competing numerical alternatives or as happily co-existing points on the x-axis of a graph

3 *Paradigmatic ambiguity,* such as Barwell's (2005) example of children perceiving thin plastic objects either as idealised two-dimensional shapes (as the manufacturer may have intended) or as three-dimensional objects of very slight thickness. In this case, we see different shared assumptions operating in a particular community of practice (Wenger, 1999). An algebraic example would be solving a pair of linear simultaneous equations in two unknowns, where the student may shift, perhaps imperceptibly, from treating the letters as unknown values to variables that (in each separate equation) can take any real value, enabling the graphs to be drawn and the point of intersection to be found. Mason and Pimm (1984) describe ambiguities between fractions and rational numbers and many writers have discussed the pregnant ambiguity of an expression such as 3 + 2 to represent both the process ("adding two numbers together") and the product as an object in itself (Gray & Tall, 1994).

4 *Definitional ambiguity,* where there is more than one way of interpreting the meaning of a mathematical term. For example, the word "radius" can represent the line segment itself or its length, "the length of the radius" being unnecessarily cumbersome – tautologous, even – in the UK, although it is common in the US to say, for instance, "the measure of the angle is 60° " rather than just "the angle is 60° " However, too-close associations, for example of a letter "*c*" and "a chair", are well-known to lead to *letter-as-object* difficulties, such as "chair = 4 legs" being coded as $c = 4l$ and then, when the *number* of legs (l) is 8, getting the answer "32 chairs" (Küchemann, 1978).

Ambiguities are to do with assumptions and perspective. The well-known Indian story of the elephant and the blind men (Saxe, 1963) describes how they each touch a different part of the elephant's anatomy and so form different, contradictory views of the nature of the animal. The all-seeing teller of the story views the complementarity of the conflicting observations which is hidden from any one individual. To become enlightened, the blind men need to exchange places with each other and experience other perspectives. Ambiguity can be seen as an opportunity to experience diverse viewpoints simultaneously. Johnston-Wilder and Mason (2005) have written, for instance, about the ambiguity between recognising a relationship and perceiving a property

In mathematics education, ambiguity has generally suffered a bad press, being lumped together with misconceptions and misunderstandings as something to be circumvented at all costs Hence, for example, the ubiquity in schools of mnemonics such as BIDMAS or PEMDAS to standardise the order of arithmetic operations. Avoiding ambiguity frequently involves creating rules to specify that "when we say X we mean Y", which can hamper opportunities for discussion. Referring to the rather dismissive view of ambiguity presented by the UK's *National Numeracy Strategy,* Barwell (2003) has written:

> Ambiguity forms an important discursive resource in school mathematics discourse, and perhaps in all mathematics discourse It is the potential for ambiguity inherent in all language, that allows students to investigate what it is possible to do with mathematical language, and so to explore mathematics itself. If, as suggested by the *National Numeracy Strategy,* all ambiguity is 'sorted out' as soon as it arises, valuable opportunities for students to learn the subtleties of mathematics could be lost [...] Rather than 'sorting out' ambiguities, teachers should see them as opportunities for mathematical exploration. (pp. 4-5)

If teachers specify mathematical tasks and definitions too tightly, they leave students little room for manoeuvre, restricting their freedom to explore interesting tensions and possibilities. Sometimes an ambiguity can be quickly resolved by providing additional information but, where the ambiguity is potentially productive, the dilemma, the tension and the contrast is lost and the energy is dissipated When an ambiguity is destroyed carelessly, students may end up knowing more, in a narrow sense, but nonetheless are somehow poorer for it, since the opportunity to negotiate meaning has been snatched away

Case study: surface area and volume

Potentially productive ambiguities can arise in many situations in the mathematics classroom while students are engaged in tasks relating to different areas of the subject. It is well-established that students of many ages experience difficulties working in three dimensions (Bishop, 1980), frequently exhibiting varied understandings of volume, capacity and surface area (Potari, & Spiliotopoulou, 1996; Tirosh & Stavy, 1999), for instance, and a reluctance to employ spatial visualization (Presmeg, 2006) Children are known to interpret the words "hollow" and "solid" in nontechnical and inconsistent ways (Jones, 1984). In the lesson discussed below, however, the at times heated discussion between six students springs from definitional ambiguity relating to several features of these terms Having definitions-in-progress affords an opportunity that would be removed if the students had finalized, formal definitions of surface area and volume from the start.

The students in question were aged 14-15 and were used to working in informal ways, where ideas flowed freely and conjectures and were made, tested and modified in an atmosphere that was, for the most part, perceived as accepting and friendly. The situation was observed opportunistically and I recorded the students' comments immediately after the lesson and wrote up the incident more fully later the same day

One group of six students had been working on the following question:

> Find the total surface area of a solid hemisphere of radius 5 cm.

Most of the students had calculated $4 \times \pi \times 5^2$ for the surface area of a whole sphere and halved it, obtaining 50π cm^2, a value that they had noted was exactly two-thirds of the correct answer, 75π cm^2. Attempts to isolate the error had failed and the consensus when I arrived at the group was that the published answer was incorrect.

I asked the students to visualize a solid hemisphere and to describe its surface, which led to a realization that the base contributed an additional πr^2 to the surface area, apparently clearing up the problem. But in fact this marked the beginning of a new discussion: [2]

Adam: So we were right for a *hollow* hemisphere?

David: Yes, we were doing hollow but it said "solid" Read the question!

Kate: No, because if it's hollow then it's got an *inside* as well. So you've got $2\pi r^2$ on the outside and another $2\pi r^2$ on the inside, so it's $4\pi r^2$ in total.

Adam: So that's the same as the surface area of a *solid sphere*. That's weird.

A particular image for surface area had been current within the class: you dip the object into a vat of paint and however much sticks to the surface is the surface area of the object. This was an extension of the view that the area of a two-dimensional shape is the amount of ink needed to colour it in, whereas perimeter is the amount of ink needed to draw around the edge. But in two-dimensional space we had never considered turning the shape over and counting the area of the *back* of the surface as well. In three dimensions this possibility now seemed unavoidable to Kate, presumably because now that the surface was no longer flat, its "other" side was more apparent The ambiguity here is between surface area of the outside only or of the outside and the inside together

David: But if you're counting insides, then a hollow sphere would be $8\pi r^2$ because that's got an inside as well

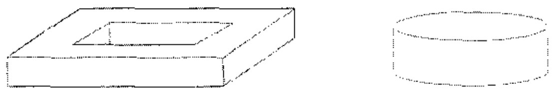

Figure 2 (a) a rectilinear doughnut (b) a cylinder

At this point, some students felt that the inside surface area would be slightly smaller than the outer surface area; others that the difference would be negligible; and others that they were exactly the same. (Does surface have any thickness?) But Kate had a different objection:

Kate: I don't think that's right. You never count insides, because if it's a whole sphere then you can't even tell if it's got an inside or not, because you can't see in it. And the paint doesn't get to the inside

David: It could be see-through, so you could tell, or you could weigh it or something, or you might know because of how you made it or what you made it out of.

I envisaged a hollow sphere with an infinitesimally small hole and the group were divided over whether the surface area of that would be $4\pi r^2$ or $8\pi r^2$ (or $4\pi r^2$ or $8\pi r^2$ minus a tiny bit). Can molecules of paint get through an infinitesimally small hole?

The discussion then shifted to a hollow *cylinder* The feeling now was that you *definitely* had to count the inside surface, so most now agreed that the surface area of a hollow open cylinder of radius r and height h was therefore $4\pi rh$ rather than $2\pi rh$ I reminded the group that when they had recently calculated the surface area of rectilinear doughnut shapes (see Figure 2) they had indeed included all the exposed surfaces.

Sarah moved the discussion towards volume Other students suggested that volume is "easy" - you can just halve it for a hemisphere, "end of story".

David: It's like with area and perimeter; area's easy because you don't have to worry about the edge when you cut something in half. Volume's the same - it doesn't matter if it's hollow or solid, the volume is the same.

Sarah: But that's stupid. If something's hollow then it hasn't *got* any volume!

This comment implied that the volume of a hollow hemisphere is zero. Sarah explained that she was envisaging placing the object in paint and looking at the volume of paint displaced - a hollow hemisphere would displace nothing, whereas a solid hemisphere would displace $\frac{2}{3}\pi r^3$. So there was ambiguity regarding volume too - does volume relate to material or to space?

Adam: But then a hollow sphere is exactly the same as a solid one, because it's empty but the paint can't get in

Ann: It might float - then it wouldn't displace any water

It is interesting to consider the implicit assumptions employed here. A hollow hemisphere is tacitly being taken to have zero (or negligible) thickness, so that when submerged it displaces no water, since the volume of a two-dimensional surface is zero. Similarly, does Ann think that floating bodies displace no liquid? Or does she suppose that a floating hollow sphere has zero mass, so it can float without sinking at all into the water?

There was much confusion between capacity and volume, with other students interpreting volume as the quantity of liquid that the object could hold

David: A cup doesn't have any volume if it's empty.

Chris: Yes it does, it's full of air.

David: Then it's the air that has the volume; the cup still hasn't got any volume, except the bit that it's made out of.

Chris: If it hadn't got any volume you wouldn't be able to put anything in it!

Ambiguity between volume and capacity (a word no-one used) meant that capacity depended for some on whether the object had a "bottom": a hollow cylinder could not hold a liquid, so had no capacity, although it was capable of containing a fixed volume of a gas, such as air, or a stackable solid. But what if you stood its end on a table - then you could fill it with something like sand, couldn't you?

Adam: Volume is how much space something has inside

Chris: Well then the volume of the solid one is zero - because it's got no space in it.

Everyone seemed to disagree with this, including Chris

Kate: Volume seems to mean two completely opposite things; either how much room it has in it or how much room it *doesn't* have in it!

I commented that most of an atom is empty space, so even "solid" things are almost entirely empty and so we are perhaps always working out the volume of mainly empty space

Conclusion

I have no doubt that this was an enjoyable and valuable discussion, but it was possible only because of ambiguity with regard to at least five matters:

1. Is the concave side of an open surface part of the surface area?
2. Is the interior surface of a closed shape part of the surface area?
3. Does the volume of a closed object include the space inside?
4. Is the volume of an open object the same as its capacity?
5. Can an object have capacity if it cannot contain a liquid?

Had there been an attempt to resolve these definitional ambiguities at the outset, the discussion would not have taken place as it did – perhaps not at all. Settling these matters by laying down watertight definitions at the start would have killed the episode (Morgan, 2005). Putting this more positively, ambiguity permits variety of thought and expression and allows (forces, even) alternatives to be considered, providing students with the opportunity to probe mathematical structure. Barwell (2003) has commented:

> Once some degree of ambiguity is constructed [...] a space opens up for the students to explore [...] In considering the role of ambiguity in mathematics classroom interaction, therefore, the aim is to understand how ambiguity arises for participants, how they deal with it and what it does in relation to the mathematical (or other) work of the discussion. (p. 4)

Ambiguity is necessary for ideas to move forward because it creates an instability in what is currently known that allows the formation of new knowledge. This relates to a fallibilist approach to knowledge and truth and is essential for the sort of fallibilistic pedagogy exemplified here (Ernest, 1999). It is vital for mathematical concepts to remain negotiable at the social level if discussion is to be genuine and meaningful. Tall and Vinner (1981) warn of the dangers of students relying exclusively on imprecise "concept images" that they have built up for themselves against more formal and accurate "concept definitions". Their idea of progress in learning is for the concept image to adjust increasingly to encompass more of the implications of the concept definition, so that instinctive responses more closely match reasoned conclusions. This means that, in the ideal, concept images are constantly shifting in the direction of greater mathematical rigour. Furthermore, it can be argued that in many cases this shifting process itself is more important than the final endpoint of a formally stated result. Learning cannot be measured simply by the number of concrete results mastered, but rather by the depth and quality of the mathematical thinking involved along the way.

The business of tightening up a concept image may be of more mathematical significance than the "final" definition arrived at, which could justify teachers in deliberately prolonging a productive ambiguity. By actively avoiding ironing out problems too soon, teachers would be deliberately delaying their students' arrival at mature concept definitions, prioritizing the process over the product. Teachers might do this by provoking their students to think about alternatives, or by playing devil's advocate, and this could afford students with increased opportunities to reason through the implications of their partially formed concept images

When a student arrives at what seems like an all-encompassing mathematical definition or theorem that appears to allow no room for manoeuvre, that particular mathematical journey is over. Students need not always be rushed to that point. On the one hand, firm understandings enable us to build and grow further ideas, which is one way in which the subject develops; on the other hand, fixed ideas take away some of the opportunities to debate and design other possibilities along the way.

For me, the episode discussed above had value beyond the immediate consideration of surface area and volume. The questions arose from the students rather than from the teacher and the discussion and the thinking involved displayed impressive mathematical reasoning. To dismiss the difficulties described as "just definitional" or as arbitrary rather than necessary (Hewitt, 1999) would be to miss the point. Adam's comments suggest a searching for definitions that will facilitate communication, while the other students tended to discuss the contextual behaviour of objects. Definitions matter in mathematics, but it also matters a great deal how they are arrived at in the classroom. It is important to recognise historically the social processes which led to their construction in response to questions that human beings have raised. Much is arbitrary in the sense that it could conceivably have been otherwise, but nothing is arbitrary in the sense that it came about for no reason.

The mathematics teacher is responsible for inducting students into a mathematical community that takes certain things for granted, yet I would much rather students worked with slightly non-standard versions of definitions that are meaningful to them and which they have arrived at by mutual thought, rather than with more typical definitions imposed on them in a more dictatorial manner. The Nobel-prize-winning physicist, Richard Feynman (1985) describes how as a teenager he developed his own idiosyncratic notations for school mathematics:

> I didn't like the symbols for sine, cosine, tangent, and so on. To me, "sin*f*" looked like *s* times *i* times *n* times *f*! So I invented another symbol, like a square root sign, that was a sigma with a long arm sticking out of it, and I put the *f* underneath. For the tangent it was a tau with the top of the tau extended, and for the cosine I made a kind of gamma, but it looked a little bit like the square root sign. [...] I thought my symbols were just as good, if not better, than the regular symbols - it doesn't make any difference what symbols you use - but I discovered later that it does make a difference. Once when I was explaining something to another kid in high school, without thinking I started to make these symbols, and he said, "What the hell are those?" I realized then that if I'm going to talk to anybody else, I'll have to use the standard symbols, so I eventually gave up my own symbols

Although Feynman gave up inventing new symbols for well-established functions, throughout his career he established an unparalleled reputation for originality and creativity (Gleick, 1994). Definitions may be provisional and pragmatic, but some agreement is necessary for collaborative work to develop. However, I believe that the productive nature of many mathematical ambiguities is enough to justify valuing and even conserving them for a while. Learning how to harness mathematical ambiguity for pedagogical benefit will require deeper engagement with the factors that make ambiguities more or less productive for students in particular situations.

Many questions remain. Is ambiguity equally productive in all areas of school mathematics? Could it be significant that Barwell's example (2005) and the one reported here both relate to geometrical topics? It would be interesting to try to locate similar instances in more algebraic and numerical areas. Are the different kinds of ambiguity listed earlier equally relevant to different mathematical areas? The overriding question is: To what extent, under what circumstances and in what ways can ambiguity be beneficial to learners? The incident recounted here was serendipitous; the pedagogical challenge is to explore ways in which productive ambiguity can be noticed and even planned for and purposefully exploited for the learning of mathematics.

Notes

[1] The F in the second example might more correctly be written as an E#, so that the interval is a diminished fifth rather than an augmented fourth

[2] All students' names are pseudonyms

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