MATHEMATICS-FOR-TEACHING AS SHARED DYNAMIC PARTICIPATION

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As researchers, our interest in teachers' disciplinary knowledge is oriented by what might be called 'big' questions: *What is mathematics? What is learning? Why teach mathematics?* In an educational culture that values pragmatic action and measurable results, these questions often appear to take a back seat to more immediate ones: *How can students do better on their next standardized test? What curriculum standards need to be covered?* Yet we have noticed that even teachers who are reluctant to take on big-picture questions will tackle deep underlying issues when they are framed in terms of specific curriculum content – *What is a number? What is multiplication? What is a function?* In fact, in our experience most teachers are curious and enthusiastic about topics such as the emergence of mathematics and its complex structure when these issues are integrated with curricular topics.

We frame this article with an account of an extended engagement of experienced teachers with the concept of multiplication. We use the account as a reference point for discussing participatory dimensions of teachers' knowledge of mathematics, and for outlining our current thinking about how to structure mathematical experiences for teachers.

Aspects of teachers' mathematical knowing

Several strategies for characterizing teachers' disciplinary knowledge have been developed in recent years. Perhaps the most common distinction in the literature is between process (being *mathematical*) and product (knowledge of *mathematics*). The categories reflect two complementary goals of mathematics pedagogy. On the one hand, teachers aim to instill attitudes and strategies that will enable learners to engage with unfamiliar problems in productive ways (*i.e.,* to act mathematically). On the other hand, such productive engagements are clearly contingent on achieving a certain level of mastery with established tools and insights (*i.e.,* the mathematics). We prefer to treat the distinction between established mathematics and dynamic mathematical processes more in terms of relative stability than as separate categories.

The explicit conclusions of formal mathematics are what, we believe, most people would identify as 'math'. Indeed, early research into the relationship between teachers' disciplinary knowledge and student achievement (*e.g.,* Begle, 1979) focused exclusively on this category. In the last decade, researchers have constantly relaxed and expanded the boundaries of the domain of teachers' mathematical knowing. Ball and Bass (2000) described mathematics for teaching as a specialized type of teacher's knowledge that links content and pedagogy. They also posited that the teaching of mathematics is a form of mathematical practice, and that teachers' mathematical knowledge should be thought of as knowledge-in-action.

Extending this line of thought, elsewhere (Renert & Davis, 2008) we employ an integral framework to correlate conceptions of teachers' mathematical knowledge in four co-evolving domains: the objective domain of mathematical objects, the subjective domain of personal meanings, the interobjective domain of shared cultural interpretations, and the intersubjective domain of external systems that enfold mathematics and teaching. We argue that 'mathematics for teaching' is an open disposition towards mathematics, which entails a willingness to harmonize the competing evolutionary tensions of mathematics and teaching as they arise in pedagogical contexts. This point is developed further in this writing.

Contemporary research offers multiple perspectives from which teachers' mathematical knowledge may be understood. They range categorically from knowledge-as-static to knowledge-as-dynamic, from knowledge-as-Platonic to knowledge-as-embodied, and from knowledge-as-established to knowledge-as-emergent. Of these categories, our current study focuses on one that might be described as the figurative substrate of established mathematics – that is, the images, analogies, metaphors, and other associations that underlie, animate, and interconnect mathematical ideas.

Participating in multiplication

The case presented here unfolded over a period of two years. It involved a group of 11 middle-school teachers who gathered every month in daylong meetings to discuss and deconstruct different curriculum topics.

These ongoing meetings have been conceived as collective knowledge-producing occasions, through which mathematics educators identify, interpret, interrogate, invent, and elaborate images, metaphors, analogies, examples, exemplars, exercises, gestures, and applications that are invoked in efforts to support the development of students' mathematical understandings. Rooted in work by Davis and Simmt (2006), the term "concept study" combines elements of two prominent notions in contemporary mathematics education research: *concept* analysis and lesson *study*.

Concept analysis, which was particularly prominent in mathematics education research from the 1960s to the 1980s, focuses on explicating logical structures and associations that inhere in mathematical concepts. As Usiskin et al. (2003) described it,

> Concept analysis involves tracing the origins and applications of a concept, looking at the different ways in which it appears both within and outside mathematics, and examining the various representations and definitions used to describe it and their consequences. (p. 1)

We blend this emphasis with the collaborative structures of lesson study, through which "teachers engage in to improve the quality of their teaching and enrich students' learning experiences" (Fernandez & Yoshida, 2004, p. 2). Lesson studies are oriented toward the production of new pedagogical possibilities through participatory, collective, and ongoing engagements. In combining the mathematical emphases of concept analysis with the interactive dynamics of lesson study, our concept studies are occasions for excavating extant meanings of concepts, as well as opportunities for shared critiques and extensions of interpretative possibilities for pedagogical purposes. To this end, the embodied and figurative dimensions of formal concepts (*cf.* Lakoff & Núñez, 2000) and the situated, distributed, nested, and participatory character of knowing (*cf.* Jenkins *et al.,* 2006) are aspects of current research into mathematical knowing that figure prominently in our present concept studies.

So far, the cohort of teachers has dealt with equality, number, zero, infinity, the basic arithmetic operations, exponentiation, and function. Of these, multiplication has received the most attention from the group. Indeed, discussions of other topics have regularly gravitated to the subject of multiplication.

We structure our account of the group's engagement with multiplication in five 'layers', or 'multi-plies'. Only the first layer could be described as intentional in any structural sense. The others were emergent – unanticipated, unplanned, arising from shared interests, divergent knowings, and accidental encounters.

Layer 1: Realizations

What is multiplication? We often pose this question in our work with groups of teachers, both pre-service and in-service, specialists and generalists. Most often, two responses are given: 'repeated addition' and some manner of 'grouping'. In order to obtain deeper, more nuanced answers, we began to ask some more focused questions. We also noticed that the quality of the answers changed markedly when we gave teachers opportunities to work together in cross-grade groups and ample time to think over the questions.

Such was the case with the concept study group in this account. At the teachers' request, our second session together was focused on multiplication. It began with the usual question, which elicited the typical responses. The rest of the morning was organized around discussions of pedagogical difficulties that crop up in teaching multiplication, investigations of when and how elaborations are introduced, and analyses of teaching resources for multiplication. The end result was a listing of metaphors, images, analogies, and applications, as shown in figure 1.

In her analysis of mathematical discourse, Sfard (2008) pointed out that while abstract mathematical signifiers act as nouns in mathematical utterances, it is the perceptually accessible *realizations* of these signifiers that enable people to create substantiating narratives about them. The teachers seemed surprised at the lengthy list of realizations of the signifier "multiplication" that they were able to generate, given the basic concept under discussion. This sentiment was summarized by the comment, "Apparently we don't have a good handle on what we know yet".

The open-ended response "and so on …" at the end of the list in figure 1 is particularly telling. At this point in the concept study, there was a clear sense that there was more to be said. The teachers realized that the point of the list was not to provide an exhaustive summary of interpreta-

Multiplication involves …

- repeated grouping
- repeated addition
- sequential folding
- layering
- the basis of proportional reasoning
- grid-generating
- dimension-changing
- intermediary of adding and exponentiating
- opposite/inverse of division
- stretching or compressing of number-line
- magnification
- branching
- rotating a number line
- linear function
- scaling
- and so on …

Figure 1. A teacher-generated list

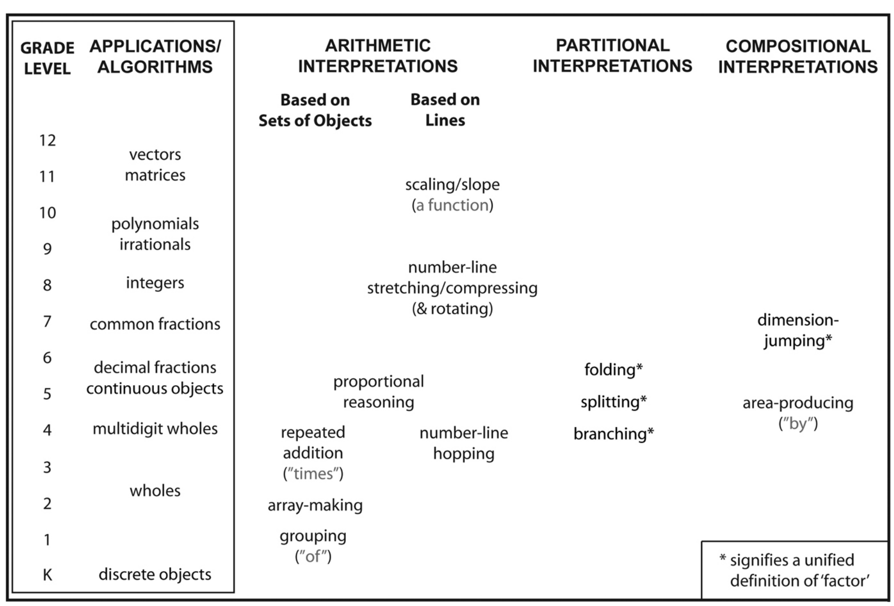


Figure 2. An evolving landscape of the concept of multiplication

tions of multiplication, but rather to indicate the range of associations that were accessible to the group on this day.

Layer 2: Structures

The next major development on the subject of multiplication took place several months later. It was prompted by Greer's (1994) "classification of situations modeled by multiplication and division" (p. 64), in which a subset of the above-mentioned realizations was categorized according to application. A few of the participants urged the group to create a similar chart to organize all of the realizations in the list. The first version of the chart took a few hours to generate and, in fact, the project goes on. The chart is modified occasionally as we revisit and elaborate the landscape of multiplication.

The vertical axis of the chart consists of points in the K–12 curricula where significant elaborations of multiplication are presented (*i.e.,* the box on the left side of fig. 2). The horizontal axis consists of thematic categories of interpretations: arithmetic, partitional, and compositional. After much debate and shifting of labels taped to the board, the group agreed on the mapping scheme depicted in figure 2.

Upon examining their mapping, participants seemed even more surprised than in the first layer with the recognition that distinct and coherent strands of interpretation are systematically developed over the K–12 experience. The different realizations of multiplication, far from being random or isolated, were organized into grander interpretive structures. The teachers acknowledged that they had in fact taken part in this systematic development before the study, without having been aware of this participation.

Layer 3: Reasoning

After a little more than a year of working together, one of the teachers brought this "small question" to the group: *Is 1 prime?* Although not the intended topic of the day, it seemed to be an important question, and one that could be addressed quickly before moving on to the day's topic.

Formal mathematics, of course, offers a definitive answer. According to the Fundamental Theorem of Arithmetic, 1 must *not* be prime, so that each number greater than 1 has a unique prime factorization. This answer, however, did not satisfy the teacher who posed the question, so she asked a follow-up: "My question is really more about why students have so much difficulty with the idea of prime numbers." The session that followed is detailed elsewhere (Davis, 2008), but aspects of it are pertinent to our discussion of layers of engagement with multiplication.

Building on earlier concept studies, the group began to explore the relevance and implications of different realizations of multiplication to the question. In the discussion, participants often framed their remarks as "If …, then …" statements, locating their comments within specific metaphorical domains. In other words, participants were consciously engaging in analogical, as opposed to logical, reasoning. Some of the results are presented in figure 3.

The contents of figure 3 should not be construed as definitive or complete. Most of the realizations of multiplication listed in figure 1 are missing from the table and even realizations represented in the table can be interpreted differently. Such is the nature of associative reasoning; it tends to open up interpretive possibilities, not shut them down. The teachers came to appreciate through first-hand experience that humans are not merely logical creatures, but association-making beings whose capacity for formal reason operates alongside their predisposition to making connections (*cf.* Lakoff & Johnson, 1999). This appreciation became the central point of the engagement, so much so that one of the teachers remarked, "No wonder the kids find that so difficult."

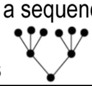
If multiplication is …	**… then a product is:**	**… a factor is:**	**… a prime is:**	**Is 1 prime?**
REPEATED ADDITION	sum (e.g., 2 × 3 = 2 + 2 + 2 = 3 + 3)	either an addend or a count of addends	a product that is either a sum of 1's or itself.	**NO**: 1 cannot be produced by *repeatedly* adding any whole number to itself.
GROUPING	a set of sets (e.g., 2 × 3 means either 2 sets of three items or 3 sets of two)	either the number of items in a set, or the number of sets	a product that can only be made when one of the factors is 1	**YES**: 1 is one set of one.
BRANCHING	the number of end tips on a 'tree' produced by a sequence of branchings (e.g., 2 × 3 means )	a branching (i.e., to multiple by *n*, each tip is branched *n* times)	a tree that can only be produced directly (i.e., not as a combination of branchings)	**NO**: 1 is a starting place/point … a pre-product as it were.
FOLDING	number of discrete regions produced by a series of folds (e.g., 2 × 3 means do a 2-fold, then a 3-fold, giving 6 regions)	a fold (i.e., to multiply by *n*, the object is folded in *n* equal-sized regions using *n* – 1 creases)	a number of regions that can only be folded directly	**NO**: no folds are involved in generating 1 region
ARRAY-MAKING	cells in an *m* by *n* array	a dimension	a product that can only be constructed with a unit dimension	**YES**: an array with one cell must have a unit dimension

Figure 3. Some analogical implications of different realizations of multiplication

Layer 4: Blends

A recent layer in the group's ongoing concept study of multiplication is that of shared reconciliation of seemingly different realizations of multiplication into unified blends. Sfard (2008) explained that the process of creating new and more powerful mathematical discursive objects, in this case, the blends, is an inevitable part of mathematizing. As the teachers engaged in this process, they repeatedly made use of the operation of *saming,* in which a single signifier is assigned to a number of objects that, so far, have not been considered to be the same in any way.

The first blend, a grid-based representation of multiplication, already was a topic of brief discussion during the first layer. As illustrated in figure 4, the grid structure pulls together several realizations – including repeated addition, array-making, and area-making – as it highlights procedural similarities in handling additive multiplicands across diverse number systems and algebraic applications.

Even though the teachers appreciated the bridging possibilities offered by grid-based multiplication, some of them expressed frustration in the model's inability to handle many of the realizations on the original list, including, in particular, scaling and number-line-based interpretations. Further analysis reveals that these realizations do not lend themselves to additive decomposition of the multiplicands.

It wasn't until some time later, at the session marking the end of our second year of concept study meetings, that an alternative blend was encountered. It occurred when one teacher noted that the number-line-stretching interpretation could be combined with the mapping-function interpretation. The mapping-function interpretation assigns to factors m and x (*i.e.,* slope and x-coordinate) the product $y = m \times x$ (*i.e.,* y-coordinate). As figure 5 shows, when mapping functions of different slopes are graphed, the vertical line $x = 1$ corresponds to the standard number-line, the line $x = 1/2$ corresponds to a number-line stretched by a factor of 2, and the line $x = 2$ corresponds to a number-line compressed by a factor of 2. These correspondences were not immediately obvious to the teachers, and required some time and deliberations to be fully realized.

The group moved on to examine the implications of the new consolidated blend for the second and third quadrants. At this point the group was deeply immersed in analogical reasoning, while making use of the new and powerful blended interpretive structure. Participants noted that the pattern of products in the right quadrants is mirrored in an "inverted" fashion in the left quadrants. In the discussion that followed, several links were drawn to other interpretations on the original list. Everyone was surprised to discover that multiplication by –1, as interpreted through the blended structure, is not a rotation about 0, but instead a compression through 0 that results in an inversion of the number-line. There was a collective satisfaction at having derived an intuitive, graphical "proof" of a mathematical result that has hitherto been enacted primarily as the inexplicable rule "a negative times a negative is positive." Indeed, this moment of creating new mathematics, which had never been encountered by anyone in the room, appeared to validate the activity of concept study.

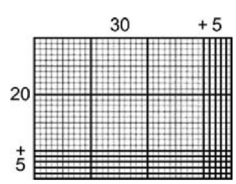
Layer 5: Participation

Given the group's positive experience in creating new mathematics, we proceeded to explore some of the broader questions of math education with the teachers. To this end, we asked the participants why they found the question *"What is mathematics?"* less than interesting and engaging than the question *"What is multiplication?"* Their responses were telling.

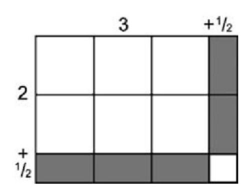
They opined that, irrespective of how it is framed, *mathematics* feels as though it has an independent, objective, extra-human existence that transcends individuals. This view was articulated in the joint remarks of two of the participants: "It feels like it's outside of us …", and "… we feel like we're outside of it." In contrast, in reflecting on the group's extended concept study of multiplication, other participants said that they "have really been able to get inside the idea," and to "feel as though [they're] really contributing to how multiplication is understood." These uses of (op)positional metaphors are striking – removed from mathematics; yet inhabiting, even responsible for, multiplication.

This difference was further reinforced when we asked the teachers to outline some of the qualities of mathematics. They generated an extensive list of adjectives that included *explicit, fixed, logical, and real (i.e., objective).* But when we asked

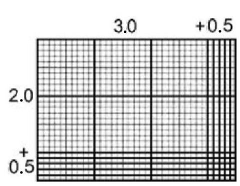
x	30	+ 5	
20	600	100	700
+ 5	150	25	175
	750	125	875



x	3	+ ½	
2	6	1	7
+ ½	1½	¼	1¾
	7½	1¼	8¾



x	3	+ 0.5	
2	6	1	7
+ 0.5	1.5	0.25	1.75
	7.5	1.25	8.75



x	$3n$	+ 5	
$2n$	$6n^2$	$10n$	$6n^2 + 10n$
+ 5	$15n$	25	$15n + 25$
	$6n^2 + 15n$	$10n + 25$	$6n^2 + 25n + 2$

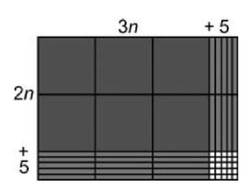


Figure 4. A grid-based blend that highlights the similarities of multiplicative processes involving additive multiplicands

them if they would apply the same sorts of descriptors to multiplication, they responded with a unanimous and resounding "No." We then asked if perhaps the antonyms of these descriptors might be used to describe multiplication. In other words, could multiplication be characterized as *implicit, evolving, analogical,* and *subject to interpretation (i.e., subjective)*? There was no consensus on this question, since our shared work seemed to show that multiplication has at least as many objective and logical dimensions as subjective and analogical ones. Still, the question gave everyone pause.

By this time, the teachers had become very cognizant of the enacted and participatory dimensions of multiplication. They seemed to be aware of their active role in creating and sustaining multiplication as a shared activity, and of the multi-plies that make up their participation in it.

Participatory dimensions of teachers' mathematical knowledge

We turn now to some of the issues surrounding mathematics for teaching that were disclosed by our concept study of multiplication. We believe that the different identifications with the concepts, as expressed by the teachers in the group, call into question many widely accepted assumptions about school mathematics. These include the roles of teachers in the production of cultural knowledge, distinctions between teachers' and students' disciplinary knowledge, and the nature of mathematical knowledge.

Teachers as participants in knowledge production

We begin with the suggestion that teachers are vital participants in the creation of mathematical possibilities. Far from being peripheral agents who passively transmit established results of mathematics, teachers give shape and substance to cultural mathematics – that is, not only to formal mathematics, but also to the range of culturally situated applications, practices, and perspectives that are enabled by formal mathematics and by other mathematical frames of reference.

The participatory dimensions of teachers' mathematical knowledge can be discerned at all layers of the foregoing concept study. In classroom practice, teachers select from and engage with many *realizations* of mathematical concepts. Their engagements condition the development of interpretive *structures* that figure throughout the curriculum. Teachers regularly use and model processes of analogical *reasoning,* which are crucial for the emergence of mathematical understandings. They *blend* and combine interpretations, in the process activating the enacted, embodied, and evolving characteristics of mathematics and mathematical understandings. In other words, teachers *participate* in the production of cultural mathematics.

These layers of enactment are largely tacit and often accidental. We saw this during our concept study of multiplication, when the teachers were regularly surprised at discovering mathematical forms and reasoning patterns that they had employed unconsciously for years. In our opinion, teachers should be encouraged to become more aware of, and intentional in, their participation in molding cultural mathematics. As discussed below, our current research suggests that conscious participation by teachers may well contribute to effective teaching.

Mathematics for teaching as a distinct mathematics

Teachers' mathematical knowledge is too extensive and too dynamic to capture in a set of resources or compress into a

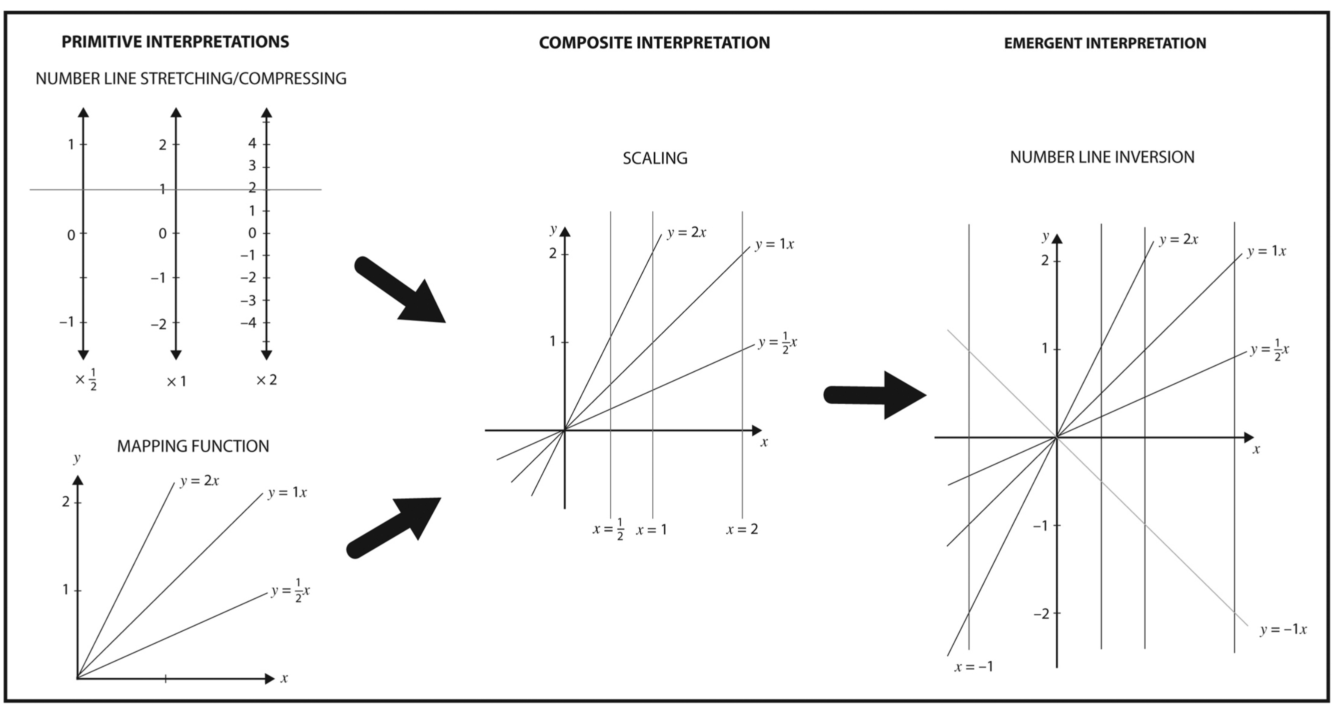


Figure 5. A graph-based meta-realization that blends linear models of multiplication

course of study. Their knowledge is perhaps best understood as an attitude toward mathematical engagement, and not as mastery of a domain of mathematics. Teachers' mathematics can be seen as a mode of being that is enacted when teachers approach a new topic, make sense of a student's error, or reconcile idiosyncratic interpretations. In quite different terms, mathematics for teaching entails awareness that personal mathematical knowing and collective mathematics knowledge are co-implicated, self-similar forms.

As the concept study of multiplication progressed, it became obvious that teachers' disciplinary knowledge could not be easily reconciled with expectations for student learning. It was clear from early on that many of the insights could not be directly worked into the grade-school curriculum. None of the participating teachers would advocate that their students' mathematical engagements be patterned closely on the examples provided.

To this end, we join with others in arguing that mathematics for teaching is a distinct branch of mathematical inquiry. It should be neither conflated with other branches of research mathematics, nor collapsed into school mathematics. At the same time, we also see the need for identifying points of commonality between teachers' and students' engagements with mathematics. Specifically, both teachers and students must be aware of the open and evolving nature of mathematical concepts. Multiplication, for example, should not be conceived as an immutable, transcendent concept, but rather as an idea that is subject to modification and elaboration as new interpretations and applications emerge. To put it differently, the images, analogies, metaphors, and interpretations associated with a mathematical concept are not merely windows into this concept, but rather constitutive elements that contribute to its shape and possibilities.

Even though many of the results obtained in concept studies cannot be applied directly to classroom teaching, we may still examine what effects they might have on student learning. How do the insights obtained by teachers who study mathematical concepts collaboratively manifest in teaching practice? This is a question that we intend to investigate in our future work. For now, we are encouraged by the enthusiasm exhibited by the participating teachers towards their subject matter.

The nature of mathematics

Returning to the issue that we raised at the outset of this article, we believe that understandings of the nature of mathematics and how mathematical truths are produced are essential elements of teachers' disciplinary knowledge. However, as noted above, even teachers who are deeply engaged in the deconstruction and elaboration of mathematical ideas can feel excluded from the subject matter, if they view mathematics as being Platonic, fixed, and objective.

By organizing concept studies around specific concepts, such as multiplication, we were able to bypass feelings of distance and exclusion. Since the teachers were in familiar territory with multiplication, and their mathematical proficiency was not an issue, the focus of the engagement shifted from technical competence to a broader discussion of the concept. In essence, this shift represents a movement from logical action to analogical reflection. By moving beyond formal mathematics to personal and shared interpretations of specific aspects of multiplication, the teachers were able to participate consciously in mathematical knowledge production. As they developed various realizations, structures, and blends, they grew comfortable with living inside the concept of multiplication as co-participants in the production of both personal knowings and collective knowledge.

We were initially surprised that the commonplace question of whether mathematics is created or discovered never arose for the group. The relevant issue for them was not the ontological status of mathematics, but instead how human knowers participate in – which is to say, inhabit, enact, embody, develop, and frame – this body of knowledge. Admittedly, as described in layer 5, most of these understandings only applied to multiplication and not to mathematics in general. Nevertheless, we were encouraged that concept studies of other topics have supported similar attitudes, and we are optimistic that future studies and deconstructions of other mathematical concepts will drive home the message that all of mathematics is participatory.

Developing teachers' disciplinary knowledge

Our experiences with the concept study group have shown that teachers' disciplinary knowledge of mathematics can be construed and cultivated as a *participatory culture*. In examining change in cultural ethos, Jenkins and colleagues (2006) outlined qualities of participatory cultures. They include:

- opportunities for expression and engagement,
- support for creating and for sharing creations,
- knowledge by members that their contributions matter, and
- social connections among members.

All of these qualities were present in the teachers' group. In this final section, we comment on some design considerations, concerns, and prospects for developing teachers' disciplinary knowledge through concept study groups.

Two factors contributed to the success of the concept study group in our example. First, the group included teachers whose classroom instruction experience ranged from 8 to 22 years. Their senior experience enabled the collective to draw on many past mathematical incidents, encounters, and anecdotes. But despite the teachers' experience in the field, it took time to reveal the group's implicit mathematical knowledge. As mentioned earlier, the group met monthly for two years in daylong sessions. The process of excavating the various layers of tacit understandings was non-linear and intermittent. Since there were no clear learning outcomes or structured resources to follow, the group worked through many false leads. Some days did not appear to be productive at all. Engaging with mathematics in this manner requires considerable time for reflection and for development of emergent possibilities.

For that reason, the ingredients necessary for concept studies cannot be pre-determined or standardized. Since concept studies draw on the tacit knowledge and specific experiences of their participants, each different study would generate its own results. For example, a concept study of multiplication in the context of pre-service teacher education may lack the depth of experience, the immediacy of classroom applications, and the range of knowledge of curriculum structures of our group of senior teachers.

Even though concept studies may not be amenable to standardization, we are more than modestly encouraged by many aspects of the experiences of the participating teachers. While we, as researchers, are motivated primarily by our interest in the mathematics for teaching, we have noticed that the participating teachers are motivated by the opportunity to effect positive transformations in their classrooms. We suspect that their motivation is supported by the group's sense of empowerment, shared curiosity, interest in exploratory thinking, and the pleasure of transacting human emotions in collectivity.

The participatory dynamics of our concept study group point to some exciting possibilities for mathematics education. Of course, for this to happen, many of the orthodoxies that pervade contemporary school mathematics must be challenged. First among these is the hierarchy of knowledge held in place by existing structures of standardized curricula, formal mathematics, and standardized evaluation. Herein lies the power of the concept study method. By taking a seemingly innocuous concept – multiplication – and asking *"What else?"* and *"So what?"* in different ways, the method allows the dynamic, embodied, and enacted dimensions of mathematics to present themselves for questioning and elaboration. Over time, as more layers are uncovered by the group, and as more knowledge is produced collectively, the radically individualistic assumptions about knowing and achievement that are common in today's math education are challenged. And thus, an opening is created for teachers to consciously participate in bringing about a more authentic mathematics for teaching.

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Which of the following lists would be best for assessing whether your students understand decimal ordering? Explain.

a) 5 7 .01 11.4

b) .60 2.53 3.12 .45

c) .6 4.25 .565 2.5

d) These lists are all equally good for assessing whether students understand how to order decimal numbers.

(from the *Learning Mathematics for Teaching Project,* University of Michigan)
