Looking for the bigger picture

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I had no idea when I submitted my paper on the notion of "crystalline concept" that the rest of the March issue of FLM (31(1)) would carry so many articles that would give me so much further cause for reflection. Overall the issue is a tribute to Ernst von Glasersfeld (1990), the great exponent of radical constructivism in mathematics education who was quoted saying that "the concepts and relations in terms of the experiential world we live in are necessarily generated by ourselves" (p. 27). The articles in general reflected this maxim. Meaney (2011) wrote of "one child's home experiences of measurement", and Samson and Schäfer (2011) presented "an exploration of figural pattern generalization." Meanwhile Gerofosky (2011) focused on the "ancestral genres of mathematical graphs" giving rich anthropological meanings to "up" and "down" and "left" and "right", expressing the view that "in working with graphing, we are not dealing with an objective reality, but with a human interpretation of the universe" (p. 19) In all these papers there are various references to the embodied basis of mathematics and the wider aspects of embodiment, gesture and semiotic activities that underlie mathematical thinking.

Oldenburg (2011) considered the specific forms of modeling as used in the workplace and compared it with the educational approach to modeling His article gave me new insight into how differentials are used to model problem situations, a practice quite different from the formalism of limit processes found in calculus courses It resonates with my own views of "a sensible approach to the calculus" in which a continuous graph is drawn with a stroke of the finger and a differentiable function is "locally straight" when highly magnified. This allows us to trace along the graph of a locally straight function to see the derivative as the changing slope of the graph itself, where dx and dy are the components of the tangent vector and continue to have this meaning in differential equations Limits are only introduced later to calculate the slope precisely either as a numerical limit or a perfect symbolic derivative when we are already able to imagine what we are looking for. Oldenburg observes that applications focus on the "covariation" of variables such as u, v, and f in the equation $1/u + 1/v = 1/f$ representing the relationship between the distances u and v of object and image from a lens of focal length f, rather than on the functional relationship between an independent and a dependent variable. I will have to re-think my "sensible approach to calculus" after this. Previously I had rejected the formulation of "covariation" between variables in the calculus because I sensed that a general function $y = f(x)$ caused y to vary directly with x, but as y varied, the variation in x need not necessarily be given uniquely. Now I sense that I must reflect further on the manner in which we humans embody and symbolize the practical and theoretical aspects of variation

Renert (2011) takes us further by attending to the wider life skills that use mathematics, in terms of sustainability of human life on the planet. This takes us from a focus on the classroom and individual conceptions of mathematics to the future development of all mankind, an enormous change in vision. It took me back to William Perry's (1970) *Forms of Intellectual and Ethical Development in the College Years,* where he traced students' intellectual growth from a simple dualism that ideas are either right or wrong, to a relativist period where other interpretations may be equally viable, and on to a far broader view where alternative views can be seen as part of a much wider vista.

The Editor reminded us in his editorial that, in 1984, I had formulated my two main research problems in mathematics education as "How do we do mathematics?" and "How do we develop new mathematical ideas?" My goals remain essentially the same. Much progress has been made in understanding what I now call the "sensori-motor language of mathematics" formulated in terms of three distinct forms of cognitive growth through conceptual embodiment, operational symbolism and axiomatic formalism, united using the notion of "crystalline concept" to highlight the necessary structure that mathematical concepts have in a given context. Creativity comes from the way in which we blend our experiences in various ways to make sense of new situations.

To make more sense of the long-term growth of mathematical thinking, I find it helpful to analyse how previous experiences affect new learning. The term "met-before" (McGowen & Tall, 2010) was introduced to refer to "a mental structure that we have *now* as a result of experiences that we met before" The crucial idea is that some met-befores are *supportive* and encourage generalization while others are *problematic* and impede understanding. For instance, the idea of an equation as a balance is supportive in representing equations with positive terms and addition, but problematic with negative terms and subtraction Embodied representations are often supportive in some instances but problematic in others and, without careful reflection to understand what is going on, can cause subtle difficulties for the learner that the theoretician may not notice. The same is true of theoretical frameworks, which subtly depend on the met-befores of those who invent them. In this way approaches to mathematical teaching and learning that appear to make strides in one direction may also cause subtle problems in others.

In the papers in the March issue, there is a great emphasis on personal construction through embodiment with links to operational symbolism, but less focus on the mechanics of operational symbolism and little on the role of formal mathematical proof. Nothing can be more inspiring than to see a student capable of looking at a recurring pattern and seeing many different ways of interpreting it as an algebraic formula. However, this situation is also part of a bigger picture. Algebra is powerful not only because it has many means of expressing a structure but also because its expressions can be freely manipulated to solve problems without the need to have a specific embodiment. In the end, symbolic fluency is vital, for without it we lack the full power of mathematical precision

In algebra, we have the power to manipulate variables representing numbers that may be positive or negative, or even complex. Representing algebra in an embodied way — say with equations as a physical balance — works well for positive quantities added together, but is less helpful for negative quantities To grow more powerful in mathematical thinking requires a focus on the essentials that operate in a more general manner

The historical allusions to Norse mythology make fascinating reading and I would not be without such a vision Yet, at higher levels, mathematics must operate in a wide range of situations where particular embodiments give meanings that may not be applicable in other contexts Consider, for example, the interpretation of a function as distance given in terms of time, and all the meanings and allusions involved as its derivative is called velocity, with velocity having a derivative called acceleration, and its derivative often called "jerk" because a sudden change in acceleration is felt as a jerk But this meaning does not fit well with simple harmonic motion, such as the case where the distance is sin*t*, the velocity is cos*t*, the acceleration is -sin*t* and the jerk is -cos*t*. In what sense is the smooth function -cos*t* a "jerk"? While some embodiments may be supportive in many ways, they may also have particular characteristics that are problematic and impede generalization

While mathematics should have meaning, such meanings should be *flexible* to take account of other situations where the mathematics may be applicable. Embodied representations are powerful in giving fundamental meanings but they usually lack the power of symbolism to formulate ideas precisely and to find exact solutions to complex problems They may also involve problematic met-befores that impede understanding in new contexts

Mathematics must harness gesture and embodiment in enactive and visual representations, but it must also develop a flexibility in the use of symbolism and, where appropriate, the later development of mathematical formalism and proof. Mathematics needs to be related to real world situations but it also needs a symbolic fluency and power of its own that sets it free to move on to previously unimaginable ideas

We therefore need to look at our own theories in more humble ways, to seek simple ways of expressing practical insights into teaching and learning. There are huge problems to address There is much for us still to do to understand the nature of mathematical thinking with its crystalline structure and longer-term supportive and problematic met-befores. We should also apply our analysis not only to the learning of students but also to the supportive and problematic aspects of our own theories.

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Modeling for life

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Almost all of us grew up mathematically with an assumption of the certainty of mathematical reasoning Given certain input, we have certain output. However, problems like climate change, the degrading changes in the sea, and health treatment decisions have an unavoidable level of uncertainty in the input, as well as some complex connections. What is causal in a multi-variable set of complex connections? (See [1] for reflections on such a fundamental shift) It is a basic characteristic of modeling in situations such as climate change, or health outcomes, that the input and the conclusions are stochastic (subject in essential ways to uncertainty) but still have an appropriate form of reliability. This characteristic leads into probabilistic reasoning, a new topic not yet well supported in the curriculum or in the preparation of teachers

As Renert (2011) notes, transforming action on such topics (and on others), also requires essential learning about modeling. Students need to practice a form of problem-based learning and simulation with multiple variables In this approach, students and teachers search for mathematics sufficient to run a simulation of changes and then make some sense of why or why not to consider information from models as reliable As Renert notes, this will require the use of modeling software, often in a black box form. This focus on modeling represents a shift from Platonist, deterministic, deductive, proof-based reasoning, towards modeling-based and stochastic-based reasoning This shift positions applied mathematics as a key defining experience of current mathematics, in place of an approach to mathematics education which takes pure mathematics as the ideal to be emulated by classrooms full of students, few of whom would become users of formal proofs.

Modeling will require the essential use of technology, and data for which we do not have even a good approximating formula. This work can be supported by information presented and reasoned about visually, rather than being immediately transferred to algebra. A move towards qualitative reasoning

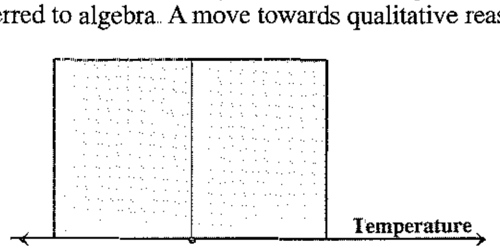

Figure 1. Simplified representation of a temperature distribution (vertical height represents number of days at a given temperature)