Visual Imagery and School Mathematics*

KEN CLEMENTS

The conclusion of this article will appear in the next issue

1. Introduction

What kinds of thought processes underlie human creative acts of the highest and most creative order? R. N. Shepard [1978] has argued that a number of highly original and significant creations of the human mind have been produced by a mode of thinking which was essentially nonverbal, involving internal representations which could best be described as images of a largely spatial, and often visual character. Shepard provides an impressive list of creative scholars whose most outstanding achievements have been the result of highly visual thinking: Einstein, James Clerk Maxwell, Michael Faraday, Hermann von Helmholtz, Francis Galton, and Friedrich A. Kekulé are on the list, as is the mathematician Jacques Hadamard. Hadamard [1945] insisted that words were totally absent from his mind when he really thought, and that they remained absolutely absent from his mind until he came to the moment of communicating the results in written or oral form. With difficult mathematical problems Hadamard claimed that even algebraic signs became "too heavy a baggage" for him and he had to rely on "concrete representations, but of a quite different nature — cloudy imagery".

Of course, Hadamard is only one of many great mathematicians who have claimed to rely heavily on visual thinking. J. F. Petrie, who discovered the properties of the skew polyhedra and polygons bearing his name, manifested an early ability in periods of intense concentration to answer questions about complicated four-dimensional figures by "visualizing" them [Coxeter, 1948, p. 31]; Alicia Boole Scott, whose father, George Boole, died when she was only 4, was inspired, at the age of 18, by a set of wooden cubes, and her reflections on those led her to an extraordinarily intimate grasp of 4-dimensional geometry which enabled her to determine, by purely synthetic methods, the sections of the 4-dimensional polytopes — well before descriptions of these, determined by more orthodox methods, were published [Coxeter, 1948, p. 258]; like Newton and Cardan, a majority of mathematicians responding to a questionnaire by Howard Fehr reported that they had solved mathematical problems in their dreams [Shepard, 1978].

(I begin a new paragraph, now, in order to avoid accusations that I regard myself as a "great" mathematician.) I recall that at the age of 15 I had a dream about an elementary geometrical problem which my teacher asked me to solve the previous day; the problem, which would be well-know to most teachers of secondary mathematics, consists of (a) proving from the information given in Figure 1 that "two sides of any right-angled triangle are parallel", and (b) identifying the fallacy in the proof.

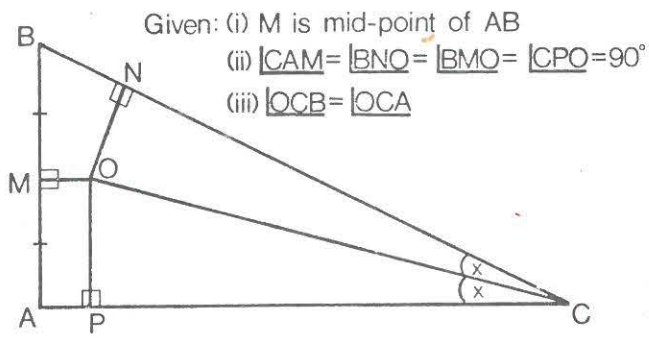

Figure 1

Before going to bed I had drawn an "accurate" version of Figure 1, and, by so doing, discovered the fallacy. In my dream I asked myself what would have happened if the line segment MO had gone to the left, rather than the right of AB, and a perpendicular had then been constructed from O to CB produced; I imagined the new diagram and proved, once again, that two sides of a right-angled triangle are parallel; then I awoke and immediately felt compelled to construct on graph paper the diagram I had imagined.

Such examples raise a number of important questions for mathematics educators. To what extent is visual imagery used by school children studying mathematics; more particularly, which children, and for which topics? To what extent do children differ in their use of imagery, and what is the influence of instruction on the fostering, or inhibiting, of such use? With respect to these questions, the observations of Shepard [1978, p.155] of the recurrence of certain suggestive patterns in the childhoods of the highly visual creative thinkers whom he considered are interesting. According to Shepard, the genetic potential for visual-spatial creations of a high order seems especially likely to be revealed and/or fostered in a child who is (a) kept home from school during the early school years and, perhaps, is relatively isolated from age mates as well; (b) if anything, slower than average in language development; (c) furnished with and becomes unusually engrossed in playing with concrete physical objects, mechanical models, geometrical puzzles, or, simply, wooden cubes. Also, the desire to press relentlessly and concertedly toward the highest achievements that such high-order creativity makes possible has required the stimulus of some previous great thinker of a similar turn of mind; thus, Shepard says, "Einstein may require his Maxwell and Maxwell his Faraday". Finally,

*A paper presented at the Annual Meeting of the Mathematics Education Research Group of Australia, at Adelaide, May 1981.

and this point would seem to have special significance when it is considered how much formal schooling encourages "normal behaviour", both in and out of the classroom, it is intriguing that many visually-inclined creative geniuses, including Newton, Faraday, Cardan and Pascal, were prone to suffer mental breakdowns, aberrations, or even hallucinations.

In order to balance the above discussion it should be pointed out that Krutetskii [1976] maintains that outstanding spatial ability is *not* a necessary ingredient of mathematical giftedness (although its possession certainly influences the mathematical cast-of-mind). Also, many scientists and mathematicians do not believe that visual imagery plays a prominent role in their thought processes [Shepard, 1978, p.155]. Furthermore, Lean and Clements [1981, in press], in discussing their analysis of performances of first-year engineering students in Papua New Guinea, concluded that "there was a tendency for students who preferred to process mathematical information by verbal-logical means to outperform more visual students on both mathematical and spatial tests. Indeed, Clements [note 1], after reviewing the literatures pertaining to spatial ability, visual imagery and mathematical learning, suggested that in some circumstances imagery mediation can have detrimental effects on abstract conceptualization [see Hollenberg, 1970]. Even in terms of memory there is evidence that imagery mediation can slow retrieval during recall [Corbett, 1977], and that it does not aid long-term retention [Hasher, Riebman and Wren, 1976]. As Wicker [1978] has said, "imagery is not likely to be a magic elixir that lubricates everything in mental life".

On the other hand, one only has to read the important recent paper by Kent and Hedger [1980], which describes successful attempts by mathematics teachers in an English secondary school to foster the use of imagery in their students' mathematical problem solving, to be convinced that for far too long educators have neglected one important mode of thinking — the visual mode. As Ausburn and Ausburn [1978] have stated, "today's child is a visual child", and visual communication technologies "have tremendous new teaching and information transmitting potential". "Why aren't we making better use of them in schools? Why force students into verbal modes in classrooms?" It is questions such as these which, no doubt, have guided those involved in the Visual Education Curriculum Project of the Australian Curriculum Development Centre, and this Project's handbook *Just imagine ... learning through mental imagery: a guide for teachers* (Visual Education Curriculum Project, 1979) is specifically aimed at assisting teachers to make greater use of imagery. But, as I have pointed out elsewhere [Clements, note 1], although psychologists have talked about training to improve imagery ability ever since Galton [1883] there is little evidence of progress at either the theoretical or pedagogical level.

The present essay briefly reviews the imagery literature for the specific purpose of drawing the attention of mathematics educators to certain issues which must be considered by all persons seriously interested in the possibility of making school mathematics more "visual". Space does not permit a full discussion of any of the seven main issues which will be addressed: (i) theoretical formulations of imagery, (ii) problems associated with the externalization of imagery, (iii) measurement of imagery ability, (iv) attempts to improve imagery skills, (v) the verbaliser-visualiser hypothesis, (vi) relationships between visual imagery and spatial ability, and (vii) research into the role of visual imagery in mathematical learning.

2. Theoretical formulations of imagery

S. M. Kosslyn [1980] has recently declined to attempt to define a mental image because "it is hard to define something one knows little about", and because the word "image" may resemble a proper name and is therefore not really definable, but simply is anchored to some entity in the world. Thus, Kosslyn maintains, the word "image" will, like most other nouns, "never be satisfactorily defined"; furthermore, "physics seems to have done reasonably well in studying "electrons", although there is not to this day a precise definition of this term" [Kosslyn, 1980, pp.469-470]. These comments by Kosslyn would appear to be eminently sensible.

As White, Ashton and Brown [1977] have indicated, mental images can arise from a number of sensory modalities — visual, auditory, cutaneous, gustatory, kinaesthetic, olfactory, and organic — and there have been many studies of such aspects as colour, form or vividness, and control of images associated with different stimuli presented to different individuals, in different sensory modalities. The focus in the present paper, however, is *visual imagery*, and it can be assumed, unless otherwise stated, that the terms "image" and "imagery" mean "visual image" and "visual imagery" respectively.

In the discussion which follows, several different descriptions of imagery arising from different theories of imagery are identified. While many theories of imagery have been put forward (see the excellent chapter on "Imagery in perspective" in Kosslyn's [1980] *Image and mind*), it will be convenient, at first, to consider those theories which fall into either one of two classes — picture-in-the-mind theories and propositional theories.

PICTURE-IN-THE-MIND THEORIES OF IMAGERY

Despite Wicker's [1978] statement that the "picture in the head" idea of visual imagery belongs mainly to history, and Paivio's [1977] comment that in modern psychology no major researcher accepts the old "mental picture" view as a working theory, the notion of a mental picture as the end product of whatever processes are involved in mental activity corresponding to the perception of an object when the object is not presented to the sense organ still persists.

According to the picture analogy theory, individuals store "mental pictures" in their memories, and these pictures, which are replicas of previous sensory patterns, appear on the mind's screen in raw and uninterpreted forms [see Kosslyn and Pomerantz, 1977]. This view has been the subject of strenuous criticism, with Pylyshyn [1973] being, perhaps, the strongest critic. Pylyshyn claimed that information could not be stored in the form of unprocessed mental photographs because there would be far too many and no possible means of organizing access to them. Thus, he argued, recalled imagery is conceptual in nature and must derive from information stored in a more abstract form; therefore the images formed are not themselves the process of imagery but by-products of internal abstract representations. It should be noted, however, that Pylyshyn does not deny the existence of images; indeed, he concedes that "we cannot speak of consciousness without, at the same time, implicating the existence of images" [Pylyshyn, 1973, p.2].

Perhaps the most influential account of imagery given in the 1970s is that provided by Paivio [1971, 1977, 1978], whose dual coding theory implies that images are richly pictorial (if not perfect copies of reality). Within the rubric of the theory, Paivio proposed that a linguistic system and an imagery system are directed at the same stimuli and work more or less redundantly. Memory was said to be dual, hence images themselves were said to be stored in the mind, presumably providing a way for people to "think in pictures" without any linguistic support. The linguistic system was said to be "specialized for serial processing" whereas imagery was basically "a parallel-processing system in both the spatial and operational senses" [Paivio, 1971, p.180]. Although, in Paivio's theory, the two systems are independent of each other, in the sense that one can function without the other, they are also interconnected because nonverbal information can be mentally transferred into verbal information and vice versa. Thus, for example, verbal instructions to form images can be used as a device to assist learning.

Another version of the picture-in-the-mind notion is found in the work of Kosslyn [1975, 1976, 1978, 1980, 1981; Kosslyn and Pomerantz, 1977; Kosslyn, Pinker, Smith and Schwartz, 1979], and especially in his computer simulation model of imagery [Kosslyn and Schwartz, 1977]. In this model the picture in the mind is made up of a rapidly decaying icon that preserves two-dimensional pictorial information of how the object "really looked" on some occasion. By this view, in contrast to Paivio's, the images have no longer-term memorial capacity; longer-term information is stored as abstract representations and the images are generated out of these representations. The "mind's eye" looks at these internal displays and classifies them in terms of semantic categories (as would be involved, for example, in realizing that a particular spatial configuration corresponds to a dog's ear). In Kosslyn's proposal the same abstract format does not store and generate mental imagery and verbal information; also, the nature of abstract representations is not specified, possibly because Kosslyn believes "it is a virtue to begin with only the outline of a model, a proto model", and that at this early stage any detailed theory of imagery "is almost certain to be wrong" [Kosslyn, 1978, p.294].

A third, and increasingly influential, modern picture-in-the-mind version of imagery is the so-called "mental analogue" position which has come to be associated with the work of R.N. Shepard [see Shepard and Metzler, 1971; Shepard and Feng, 1972; Cooper and Shepard, 1973; Metzler and Shepard, 1974; Shepard, 1975; Shepard and Podgorny, 1978]. Shepard and Metzler [1971] showed that the amount of time it took subjects to compare two figures (which were either identical or mirror images of each other) was a linear function of the angle between them. They interpreted this finding as evidence for subjects imagining an image of one of the figures rotating so that it became congruent, or not congruent, with the other, and commented that the subjects "could only do this at a certain rate without losing the essential structure of the rotated image" [p.703]; further, it was claimed the mental image was rotated in mental space in a manner exactly analogous to the way corresponding physical objects would be seen to rotate. Shepard and Podgorny [1978, p.224] wrote that the similarity of such results indicate that the same kinds of internal processes are operative in both cases.

There have been several criticisms of this "mental analogue" viewpoint. Kolers and Smythe [1979] claim to have identified three kinds of analogies which can be attributed to the imagery which Shepard and his colleagues have proposed: (i) between the physical structure of an object and its mental representation; (ii) between the intermediate states of an object and its mental representation on perceived and imagined operations, and (iii) between the mechanisms or machinery underlying imagining and perceiving. According to Kolers and Smythe [1979, p.161] these three kinds of analogies are not wholly compatible, and, therefore the basis for the analogue position is not clear. Pylyshyn [1979a] has expressed a similar criticism to that of Kolers and Smythe. According to Pylyshyn it is not clear with the analogue theory exactly what aspects of the metaphorical object are supposed parallel to the image rotation case. He points out, for example, that Cooper and Shepard [1973] claimed that between imagining a figure in one orientation and imagining it in another orientation the image, like the corresponding physical object, passes through all intermediate orientations, but adds [Pylyshyn, 1979a, pp. 19-20]:

> *No one, to my knowledge, has suggested that the image must accelerate and decelerate or that the relation among torque, angular momentum, and angular velocity has an analogue in the mental rotation case. Of course it may turn out that it takes subjects longer to rotate an object that they imagine to be heavier, thus increasing the predictive value of the metaphor. But in that case it seems clearer that, even if it was predictive, the metaphor would not be explanatory (surely, no one believes that some images are heavier than others and the heavier ones accelerate more slowly).*

Pylyshyn continued his argument by pointing out that the explanation would have to appeal to the subject's knowledge of the behaviour of heavier objects rather than to the intrinsic property of images; such an appeal to tacit knowledge is, according to Pylyshyn, required in most cases of cognitive activity and "is the primary reason for preferring a propositional to an analogue account of mental processes". Thus, Pylyshyn maintains, the more carefully Shepard's mental rotation findings are examined "the more we find the informally appealing holistic image-manipulation views must be replaced by finer-grained piecemeal procedures that operate upon an analyzed and structured stimulus using largely serial, resource-limited mechanisms", and that "by the time the reduction of global phenomena such as "mental rotation of images" to a sufficiently primitive process form has been achieved the resulting model may contain few, if any, components worthy of the name "analogue"" [Pylyshyn, 1979a, p.27].

PROPOSITIONAL THEORIES OF IMAGERY

The propositional or symbolic representation theory of imagery arose out of dissatisfaction with the picture-in-the-mind theories, especially the belief that these theories did not meet Chomsky's "explanatory adequacy" [Chomsky, 1964, 1980; Pylyshyn, 1973]. Pylyshyn [1973] pointed out that a major weakness of the picture theories is that they do not explain how verbal information can be transferred, in the mind, into nonverbal information, and vice versa. This led to the search for a deeper structure of imagery, and ultimately to the proposal of a kind of mental representation which resembled neither pictures nor words, was not necessarily accessible to subjective experience, but could explain how the transformations occurred. The various propositional, or symbolic, representation theories put forward in the early 1970s [Simon, 1972; Anderson and Bower, 1973; Pylyshyn, 1973; Norman and Rumelhardt, 1975] were all based on the idea that a person's knowledge could be represented in memory by sets of propositions, and the common format of these propositions would enable both verbal information and mental images to be generated. The claim is that an adequate account of how an organism reasons would in most cases have to depict that organism as accessing some explicit symbolic encoding of knowledge about the represented domain, as well as certain rules or explicitly represented algorithms rather than as merely following causal physical laws [Pylyshyn, 1979b].

Somewhat ironically, Kosslyn and Pomerantz [1977] criticized the propositional representation theory for the same reason that Pylyshyn criticized picture-in-the-mind theories. If Pylyshyn could ask how, according to Paivio's dual coding theory, verbal information and mental images could be transformed into each other, then Pylyshyn himself could be asked a similar question with respect to the propositional representation theory: how can information be transformed from an abstract representation into verbal information or a mental image? Kosslyn and Pomerantz also maintained that it was not clear that one needed to postulate the existence of an abstract form of representation to explain the transformation from verbal information to mental images, and vice versa. Kieras [1978] has suggested, for example, that verbal and nonverbal information can both be stored together in a common format in memory; further, since this common format is not abstract, every piece of information can retain its verbal or imaginal modality.

The debate over whether visual imagery is encoded in terms of properties that are quite spatial and modality-specific or in an abstract propositional format was given new impetus when J. R. Anderson [1978] published a paper in which he claimed that barring decisive physiological data, it will not be possible to establish whether an internal representation is pictorial or propositional. According to Anderson propositional versus dual-code models can be made to yield identical behavioural predictions. Anderson, who had previously been clearly identified with the propositional viewpoint, concluded that the picture metaphor was the only current explicit interpretation of the image theory and that frequent criticisms made of the picture metaphor were not valid. One could have a viable dual code model involving picture and verbal representations. Further, the arguments for the necessity of a propositional representation were far from compelling (but so were arguments for imaginal representations based on introspections). Anderson's paper drew immediate responses by Pylyshyn [1979b] and Hayes-Roth [1979], both of whom argued that empirical procedures can be used to decide whether internal representations are pictorial or propositional. Pylyshyn [1979b, p. 383] also stated that "the considerations that lead Anderson to endorse a hybrid or a dual code view are ill-founded". Anderson [1979] replied to the Pylyshyn and Hayes-Roth articles, conceding that although their papers provided no solid bases for establishing unique identifiability, they did reinforce the possibility of non-behavioural bases from which a decision on whether mental representations were propositional or pictorial could be made.

The intensity of the "propositional versus pictorial" debate proved to be rather frustrating for psychologists and educators who, although convinced of the importance of imagery in thinking and learning, were not inclined to proceed with research involving the concept of imagery until the definitional problem was clarified. Anderson's [1978] paper served the useful purpose of suggesting that the problem of defining imagery may be insoluble; if this is the case, then those interested in the effects of imagery, and in whether imagery ability can be trained, should feel less guilty if they conducted research despite their lack of understanding of the imagery construct. As Finke [1980] has pointed out, the question of what a mental image is can be separated from the question of what types of effects mental images can have, whatever their nature may be. As I stated earlier in the present paper, we do not find it confusing, or paradoxical, to study the effects of certain physical entities (e.g. electricity) even if we cannot be sure we fully understand their make-ups, so why should we hesitate with familiar, yet elusive concepts such as imagery? Perhaps, as R. N. Shepard [1978, p.134] has suggested, we need to be "less than fully satisfied with research that is exclusively motivated by current theories of linear sequential processing or propositional structures" and not be afraid to have a closer look at those various sorts of mental images that have played "a crucial role in processes of scientific discovery". The present writer, for one, has little sympathy with any theory that would discount the testimony of persons who claim to have "rotated an image". I cannot agree with Pylyshyn's [1979a] claim, that we need a more adequate explanation of internal representations of such events.

Nevertheless, during the period 1979-1981 the debate over the nature of imagery has continued to gather momentum, with Kosslyn and Pylyshyn being particularly active. Pylyshyn [1981], after referring to the paper by Kosslyn *et al* [1979] as "the most explicit formulation of the "imagistic" (or "pictorial" or "analogical") position to date", went on to criticize Kosslyn for misconceiving what is reported in imagery. In Pylyshyn's own words [Pylyshyn, 1981, pp.18-19]:

> *What people report is not properties of their image but of the objects they are imagining. Such properties as color, shape, size and so on are clearly properties of the objects that are being imagined. This distinction is crucial. The seemingly innocent scope slip that takes* image of object X with property P *to mean* (image of object X) with property P *instead of the correct* image

of (object X with property P) *is probably the most ubiquitous and damaging confusion in the whole imagery literature.*

Using this line of reasoning, Pylyshyn proceeds to deny Kosslyn's claim that images can have spatial characteristics such as length, width, depth, shape and size, and to argue that most of the empirical phenomena involving so-called "transformations of images" is best explained by a "tacit knowledge" rather than by an "analogue" theory.

In *Image and mind* Kosslyn [1980, pp.112-173] extends his earlier theory [see, for example, Kosslyn, 1978] that visual mental images are like displays on a cathode-ray tube that are generated by a computer program (plus data). According to this theory, images have two major components, a quasi-pictorial "surface representation" in active memory and a "deep representation" in long-term memory, and Kosslyn [1980, p.153] actually gives a detailed flowchart outlining how surface images are generated from deep representations. Computer procedures are used throughout the specification of the model, and Kosslyn even reproduces two simulated images of a car that were produced by his computer program [p.154]. Kosslyn shows how his theory can explain how a person inspects and/or transforms images; and he maintains that the mind's eye can be "equated with a set of procedures that operate to categorize spatial patterns" [p. 160].

Concerning Pylyshyn, Kosslyn [1981, p.59] refers to "Pylyshyn's mistaken view of our theory", and maintains that "not only do Pylyshyn's results fail to disconfirm the theory but the theory does not even have to strain to explain them". Kosslyn also believes that Pylyshyn's view that subjects invoke tacit knowledge, which does not involve a spatial medium, when deciding how to respond in an experiment, is false. It is not possible here to give an account of Kosslyn's detailed criticisms of the tacit knowledge position, but it suffices to say that Kosslyn [1981, p.63] concludes that "Pylyshyn has presented some general metaprinciples for a theory, not a theory itself... [his] accounts seem virtually unconstrained and have no clearly specified parameters, free or otherwise".

OTHER THEORIES OF IMAGERY

The French psychologist Michel Denis, in his recent book *Les images mentales* [1979], has provided a theoretical account of imagery which differs qualitatively from the theories put forward by Paivio, Shepard, Kosslyn and Pylyshyn. Denis' review chapters make it clear that although he is both informed and respectful of imagery research and theory published in English he aligns himself with those who insist that imagery is an active constructive process. He distinguishes between figural schemata elaborated throughout an individual's life and stored in long-term memory, and the mobilization of available figural schema in a particular situation, the latter giving rise to the subjective experience of mental images.

According to Denis, representational units are stored in memory and these are activated by specialized processes. The units are componential in nature and an image is a set of figural *features* which correspond to physical attributes stemming from the individual's past perceptual activities. The features are hierarchically organized according to their probability of contributing to image formation, and within each hierarchy there is a critical categorical level *below* which the evocation of a specific image is possible and *above* which the evocation of a proper image, other than one resembling a lower level feature, is not likely [Denis, 1979, p.169]. Thus, for example, in the chain *furniture-seat-chair-kitchen chair* the critical level would appear to be located at *seat*, since it is relatively easy to form images of *chairs* and *kitchen chairs*, but not so easy for *seats* and *furniture* unless some definite context is specified. The second main aspect of Denis' theoretical stance on imagery is that activational *processes* can be applied selectively to the representational units to generate images. These processes activate just those representational units pertinent to the task at hand. Thus, for example, the words "apple pie" can evoke different images depending on whether one wants to imagine eating an apple pie or to use an imagined circular representation for solving arithmetic problems involving fractions [see Marriott, 1978; Baylor and Gascon, 1981].

In summary, Denis' theory presents the image not as an instrument of understanding but as something which preserves information by providing mental analogues both of static configurations and their transformations. While imagery can result in better recall and faster reaction times for some tasks, this is not necessarily the case for all, or even most tasks; generally speaking imagery is not an "irreducible and indispensable psychological substrate" [Denis, 1979, p.223]. Denis' view that extensive use of visual imagery might be a disadvantage on certain tasks would appear to be especially relevant to mathematics educators, some of whom hold as an article of pedagogical faith that children's conceptual understanding is necessarily enhanced whenever visual imagery is used [e.g. see Lin, 1979].

Other theories of imagery are implicit in what has come to be known as the representational-development hypothesis [see Kosslyn, 1980, p.408]. This hypothesis has three parts: (1) the type of internal representation that is utilized changes over age; (2) the ontogenetically later forms of representation are more powerful than the earlier ones; and (3) the early forms of representation are not eradicated but are supplemented by the later forms, which tend to overshadow the previously preferred modes. In particular, it is often claimed that young children rely more on imaginal representations but older people make greater use of "abstract" representations. Interestingly, such a view would appear to be supported by recent findings of mathematics educators [see, for example, Lean and Clements, 1981, in press].

The only formulations of the representational-development hypothesis which will be considered here are those advanced by Bruner, Olver and Greenfield [1966], and by Piaget and Inhelder [1971]. Bruner *et al* [1966] claim that the child initially has only "enactive" representations and can represent information solely in an enactive format; at about the age of twelve months,, however, the child begins to use imagery to represent the world ikonically, that is to say, the child can form "pictures in the head"; but these "pictures" are static and are often cluttered with unnecessary detail. Later, the child reaches the symbolic stage which is characterised by "categoricality, hierarchy, prediction, causation and modification" [p.47]. According to Bruner, development from the "enactive" to "ikonic" to "symbolic" stages occurs as a result of "disequilibration'', that is, attempting to resolve conflicts between systems of representation. However, as Kosslyn [1980, p.409] has indicated, Bruner's account of the mechanisms responsible for representational development is weak, for the concept of ''disequilibration'' was not well worked out even by Piaget, who expended considerably more effort on its exposition.

Piaget and Inhelder's [1971] formulation of the representational-development hypothesis is embedded in the larger framework of Piaget's theory as a whole [see Greenspan, 1979]. In *Mental imagery and the child* Piaget and Inhelder [1971] distinguished *reproductive* images, which correspond to known objects or events, from *anticipatory* images, in which objects or events not previously perceived are represented. They also noted that images can be immediate or deferred, and pointed out that it is only deferred images which are isolated from the sensory stimuli. Further, images can be *static, kinetic* or *transformational*. Interestingly, although developmental studies on imagery emphasize the role of action in assisting the development of image, the picture-in-the-mind view of visual imagery is always implicitly assumed in them.

Bruner *et al* [1966] and Piaget and Inhelder [1971] agree that children are not capable of forming mental images until they have passed through a stage when objects can only be represented by actions that are performed on the objects; they also agree that it is only after imagery representation comes that children are capable of thinking about things which are not directly perceived. Piaget and Inhelder maintain that images are not derived from perception but from ''an interiorization of imitation''. Also, in contrast to the views of Bruner *et al*, they emphasize that the *nature* of a person's imagery changes with age, with children in the preoperational stages being capable of ''reproductive'' imagery only, but concrete operational children being capable of ''anticipatory'' imagery. But as Kosslyn [1980, p.410] has said, Piaget and Inhelder have never made clear the meaning of their assertion that images are ''interiorized imitations'', and they have not specified how interiorized imitations operate; further, they have never specified the format or content of the image. Thus, it would appear that at no stage did they even formulate a *theory* of imagery; the same could be said of Bruner *et al* [1966].

While the main issues raised in the foregoing brief review of the difficult literature pertaining to theoretical formulations of imagery might seem to be of little relevance to mathematics educators, the increasing availability and use of visual aids by mathematics teachers, as well as the growing realization that imagery can be of fundamental importance in the teaching and learning of many topics in mathematics [Hayes, 1973; Vinner and Hershkowitz, 1980; Lean and Clements, 1981 in press], is likely to generate interest among mathematics educators in the mechanisms by which visual images are evoked. And, undoubtedly, the desire to specify these mechanisms is at the heart of the controversy over definition.

A second, and equally important, reason why mathematics educators should concern themselves with theoretical formulations of imagery is that unless they do they will not be able to take full advantage of the substantial body of research on imagery which has been carried out by psychologists, following the remarkable return to favour of imagery research in the 1970s. Mathematics educators will continue to give the impression of naiveté in matters related to research concerning imagery and spatial ability unless they become better acquainted with the relevant psychological literatures.

3. Problems associated with the externalization of imagery

One of the main problems in imagery investigations is that of finding an appropriate research methodology. In the past introspective and retrospective reporting have often been used [see, for example, Hayes, 1973], but in 1977 a series of articles by Nisbett and his colleagues [Nisbett and Bellows, 1977; Nisbett and Wilson, 1977a, 1977b] brought into the open the suspicion which many psychologists had of data derived from such procedures. Nisbett and his coauthors gave detailed accounts of experiments which, they claimed, demonstrated that self-reports of mental activity are either completely inaccurate or no more accurate than guesses made by observers equipped solely with information about the public features of the situation. Like other cognitive and experimental psychologists [e.g. Bloom and Broder, 1950; Miller, 1962; Neisser, 1967; Mandler, 1975], Nisbett concluded that conscious awareness is limited to the products of mental processes, and that the processes themselves are beyond the reach of introspection.

The reaction against introspective and retrospective data quickly became widespread. For example, the English psychologist J. S. T. Evans, in making the point that use of introspections, even as corroborative evidence, is dangerous, referred to his work with Wason [Wason and Evans, 1975] in the following criticism of Kosslyn's acceptance, in his imagery research, of introspective data [Evans, 1980, pp.293-4]:

> *It seems that he [Kosslyn] and like minds accept them if they achieve three criteria: clarity of expression, confident assertion by the subject, and consistency with the behaviour observed. In the experiments which I ran with Wason all three criteria were met on protocols that were nevertheless apparent rationalizations. Thus Kosslyn's notion of introspections as useful corroborative data is unsound.*

Evans went on to assert that there is a considerable range of evidence in social and cognitive psychology which suggests that introspections do not indicate awareness of underlying causal processes; rather, they consist of cause-effect type theories which subjects construct when attempting to explain their behaviour.

Lohman [note 2], in his comprehensive report on spatial ability, is another to argue that ''introspective reports are of limited value'' and that ''retrospective reports are even less trustworthy''. According to Lohman, introspections should always be validated against external information, and many processes, especially those which are extremely rapid, cannot be assessed by such methods; ''detailed retrospections are probably quite unreliable'', and can provide little more than a rough index of strategy [Lohman, note 2, pp.149-150].

It should not be imagined, however, that all psychologists now refuse to accept introspections and retrospections as useful data. Egan and Grimes-Farrow [note 3], Lundh

[1979], Ericsson and Simon [1980], and White [1980] are among influential psychologists who have defended the value of such data. Ericsson and Simon's taxonomy of verbalization procedures, as summarized in Table 1, would appear to provide a theoretical foundation for discussion on the value of verbal reports as data. The two dimensions of Table 1 represent two major distinctions, namely the *time* of verbalization and the extent to which the verbalization is a direct articulation or explication of relevant information stored in memory or is such that the stored information has been used by the subject as input to intermediate processes, such as abstraction and inference.

Table 1

A classification of different types of verbalization procedures as a function of time verbalization (rows) and the mapping from heeded to verbalized information (columns) — after Ericsson and Simon [1980].

	Relation between heeded and verbalized information			
		Intermediate processing		
Time of verbalization	Direct one to one	Many to one	Unclear	No relation
While information is attended	Talk aloud Think aloud			
While information is still in short-term memory	Concurrent Probing	Intermediate inference and generative processes		
After the completion of the task-directed processes	Retrospective Probing	Requests for general reports	Probing hypothetical states	Probing general states

Ericsson and Simon point out that in studies in which probing and/or verbal reporting takes place at the end of an experiment, and subjects have not been made aware that they will subsequently be asked to report on their processes, the reporting task cannot affect those processes; of course, if subjects know they will have to give verbal reports on their processes, it is possible that this very knowledge will affect the processes they use. Clearly, however, the greatest concern about possible effects of verbalization on the course of the cognitive processes arises when the verbalization is concurrent with the task performance. It is in this sense that it could be argued that a subject's verbal reports provided *soon after* the task has been completed are likely to give a more faithful reflection of processes used *naturally* than concurrent verbal reports, and this is especially the case if the cognitive processes are of short duration.

Of particular relevance to the present paper is Ericsson and Simon's discussion of verbalizations of visual encodings. An instruction to describe a visual scene verbally presumably requires a verbal recoding of the picture, which might require extensive processing. Ericsson and Simon maintain that requiring subjects to verbalize explanations for tasks with complex visual stimuli may cause alterations to be made to strategies, and hence performances. Thus, for example, if subjects who had attempted a series of Shepard and Metzler [1971] rotation tasks were asked to verbalize their strategies one should not expect that the reported strategies would correspond well with the actual strategies. This would seem to present special difficulties for imagery researchers who gather data from verbal reports of subjects.

Egan and Grimes-Farrow [note 3] used retrospective reporting to show that people spontaneously adopt different mental representations for solving three-term series reasoning problems. Despite the fact that the reasoning tasks were presented in written forms, many subjects reported using concrete, visual thinking and such recoding of the stimuli might be expected to have required considerable amounts of processing. Nevertheless, Egan and Grimes-Farrow claimed that their classification of subjects based on retrospective reports was valid, with subjects giving different retrospections exhibiting different patterns of reasoning errors with their retrospections.

While it must be admitted that general conditions by which introspections and retrospections will provide valid data remain to be established, the twelve guidelines put forward by White [1980, pp.109-110] would appear to serve as a good starting point for those intending to collect such data. The current trend among educational researchers away from experimental methodologies toward ethnographic procedures, and the great interest in problem-solving procedures, make it imperative that careful consideration be given to how studies can be designed so that verbal reports will provide valid and reliable data. Mathematics educators have only recently begun to address the problem, and then mainly indirectly under the unlikely heading of "metacognition" [see, for example, Silver, note 4]. The present writer believes that there is much scope for creative development of methodologies which go beyond introspections and retrospections and produce better quality verbal data. Efforts of this kind might prove to be especially valuable for imagery research; the instrument developed by Suwarsono [and used by Lean and Clements, 1981, in press] represents one mathematics educator's response to this need.

Imagery researchers have created many ingenious methods by which mental representations can be empirically studied without subjects having to provide verbal data. Space does not permit discussion of such methods here; it suffices to say that Shepard and his co-workers have been especially productive in this regard, and recent papers by Pylyshyn [1979a] and Sternberg [1980] provide descriptions of other very elegant experiments.

4. Measurement of imagery ability

Given that the notion of imagery, and visual imagery in particular, is not well defined, it is hardly surprising that the expression "measurement of imagery ability" is not easily interpreted. Is there such a thing as imagery *ability*, and if so, what is it and how can it be measured? [See Mandler, 1976]. In Galton's [1883] classic study in which he asked people to visualize their breakfast tables and to describe the clarity of their images, large individual differences were reported, not only in the vividness of imagery but also in its controllability (i.e. the degree to which a subject could summon and direct it at will). Subsequent research has, almost invariably, emphasized these aspects of visual imagery [see Richardson, 1977a; Evans, 1980], and attempts to measure "vividness" and "controllability" of imagery in individuals have been made. Such attempts have necessarily involved the provision of *operational* definitions of imagery ability.

At least four different operational definitions of imagery ability have been commonly used: (i) objects or pictures of objects known to evoke varying levels of vividness of imagery have been shown to subjects, and the vividness and/or clarity of imagery evident in responses (sometimes verbalizations, sometimes marks on questionnaires) scored; (ii) words which normally evoke varying levels of imagery have been read or shown to subjects, and subjects' responses scored; (iii) subjects have been asked to answer a number of specific questions concerning their use of imagery in specific situations, and responses scored and summed (sometimes Likert, semantic differential, or other scales have been employed); and (iv) various tests of visuo-spatial ability have been given to subjects and it has been assumed that imagery ability is directly proportional to some linear combination of scores on these tests.

Richardson [1977a] reported an important series of studies in which subjective and more objective measures of imagery were obtained, the prime purpose of the studies being to investigate problems associated with the measurement of imagery. A comprehensive battery of 41 tests and self-report measures was administered to 427 first-year psychology students over a period of twelve months (some students did not take every test). An indication of the range of performance variables is given below.

1. Performance tests of imagery vividness: picture memory task [Lumsden, 1965]; position memory for pictures test; memory for designs [Graham and Kendall, 1960]; imagery response times (concrete and abstract nouns); gaze break.
2. Self-report imagery scales: Betts' QMI test [1909]; the ways of thinking questionnaire [Paivio, 1971]; an habitual mode of thinking scale, several modality-specific vividness of imagery scales devised by Richardson; a modification of Gordon's [1949] control of imagery questionnaire; an imagery control task.
3. Spatial ability tests: the Gottschaldt figures test (Form B); Necker Cube fluctuations; cutting a cube test; hidden figures test; cube comparisons, paper folding, and revised Minnesota paper form board tests.

Richardson's battery of tests also included general ability tests and personality questionnaires.

Factor analysis of results revealed an "imagery vividness factor" on which the three main self-report imagery measures (Betts' QMI, the ways of thinking questionnaire and an overall vividness of imagery scale) loaded strongly. Furthermore, no evidence was found to justify the use of any objective (performance) test as an alternative to self-report tests in the measurement of vividness of clarity of imagery. In particular, tests of spatial ability, which loaded on a separate factor, appeared to be inappropriate for this purpose. Richardson [1977a, p.42] concluded that "self-report measures must necessarily serve as the initial criterion against which other more objective behavioural and physiological measures can be validated".

Clearly, a major source of difficulty with attempts to measure imagery ability is that relationships between the different dimensions of imagery have not been established. Thus, although there is the intuitive feeling that visuo-spatial tests should require substantial use of imagery, Richardson, Frampton [1978] and Lean and Clements [1981, in press] have all reported studies in which performances on spatial tests did not correlate significantly with scores on self-report imagery scales. These results need to be seen in the light of recent work by Egan [1979] and Lohman [note 2] which showed that individuals often use analytic rather than visual methods when attempting so-called "spatial" tasks. As Saltz [1976] has said, "if we define a child as high or low in imagery on the basis of a test, it is important to know the correlates of this test to be sure that our results are not due to confounding effects of these correlates".

Mathematics educators might be especially interested in the instrument developed by Suwarsono [see Lean and Clements, 1981, in press] which is intended to measure an individual's *preference* for using visual imagery in mathematical problem-solving situations. They might also be interested in Lean and Clements' finding, which has been independently confirmed by Suwarsono himself (Suwarsono's sample consisted of seventh-graders in Victoria whereas Lean and Clements' consisted of first-year engineering students at a university in Papua New Guinea), that preference for use of imagery in mathematical problem-solving is not associated positively with performance on mathematics or spatial tests.

Notes

1. Clements, M. A. *Spatial ability, visual imagery and mathematical learning*. Paper presented at the annual meeting of the American Educational Research Association, Los Angeles, April 1981
2. Lohman, D. F. *Spatial ability: a review and re-analysis of the correlational literature*. Stanford: Aptitude Research Project, Stanford University School of Education technical report, No. 8, 1979
3. Egan, D. E. and Grimes-Farrow, D. D. *Differences in mental representations spontaneously adopted for reasoning*. N.J.: Bell Laboratories, 1979
4. Silver, E. A. *The role of memory in mathematical abilities*. Paper presented at the conference on mathematical abilities at Athens, Georgia, 12-14 June 1980

[*A full bibliography appears at the end of this article, which will be completed in* FLM Vol. 3, No. 3 (March 1982)]