Cognitive and Developmental Psychology and Research in Mathematics Education: some theoretical and methodological issues *

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* Based on a talk to the Canadian Mathematics Education Study Group at Queen's University, Kingston, June 1982

The most difficult problem in an interdisciplinary field like mathematics education is the problem of developing new concepts and new methods enabling us to approach it scientifically. Psychologists tend to use their own framework such as associative learning and conditioning (for empiricists), logical structures (for Piagetians), information-processing (for computer-oriented psychologists), psycholinguistics for others ...

On the other hand, mathematicians and maths educators tend to remain satisfied with their knowledge of mathematics and general theories of education (mainly about algorithms, demonstration and heuristics) and evaluation.

It is a scientific challenge of our time to promote the study of learning and teaching mathematics as a well-defined and interesting scientific field of its own, not reduceable to mathematics, to psychology, to linguistics, to sociology or to any other science. This does not mean that research in maths education (didactics as we call it in France) can be blind to other fields: it must be widely open to ideas coming from other disciplines. But the time has come to delineate its identity. This requires the analysis of the different contents of mathematics, in their specificity, and the empirical study of their learning and teaching, in a way that will take account both of the long-term growth of knowledge in children and adolescents, and the short-term change of conceptions in newly-encountered situations.

I will try to develop and illustrate six main points:

1 – an interactive conception of concept formation;
2 – a developmental approach;
3 – theorems in action and invariants;
4 – conceptual fields;
5 – representation and the problem of adequacy between signifier and signified;
6 – problems of methodology.

1. An interactive conception of concept formation

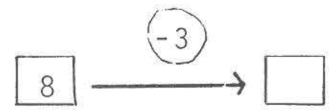
Not only in its practical aspects, but also in its theoretical aspects, knowledge emerges from problems to be solved and situations to be mastered. It is true for the history of sciences and technologies; it is also true for the development of cognitive instruments of young children (organizing their representation of space, symbolizing, categorizing objects ...). It should also be true for education, especially maths education. This is not actually the case, and it is not an easy thing to achieve: the most general tendency in maths education is to teach algorithms either independently of problems, or by referring them to only a narrow range of problems.

A first priority for research is to collect, analyze and classify, as exhaustively as possible, problem-situations that make a mathematical concept functional and meaningful, so as to be able to use a larger variety of situations in teaching, and to get students to meet other relationships and other questions than those they are accustomed to.

Conceptions, models and theories of students are shaped by situations they have met. There may be big gaps between children's conceptions and mathematical concepts: for instance, the concept of fraction may refer to so limited a set of situations for students that they fail to grasp the reasons why fractions are powerful tools.

I will take examples mainly from the field of "additive structures" in order to give more unity to the different points I will raise in this paper. But the same ideas can be raised in other fields.

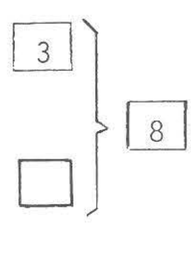
The very first conception of subtraction for a young child is a "decrease" of some initial quantity, by consumption, loss or sale, for instance. Let us represent it by an arrow joining the initial state to the final state.



Example 1: John had 8 sweets, he eats 3 of them. How many sweets does he have now?

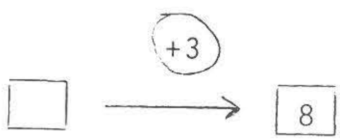
It is not straightforward, with such a conception in mind, to understand subtraction

– as a relationship of complements



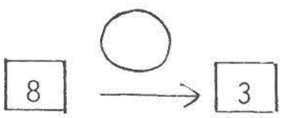
Example 2: There are 8 children around the table for Dorothy's birthday. 3 of them are girls. How many boys are there?

– as the inverse of an increase



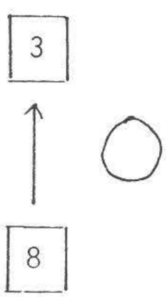
Example 3: Jane has just been given three dollars by her grandmother. She now has 8 dollars. How many dollars did she have before?

– or as a difference relationship
  - between states (included in each other)



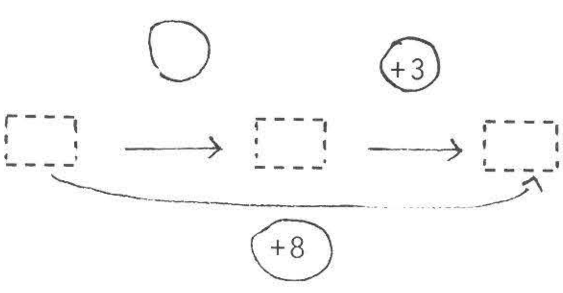
Example 4: Robert had 8 marbles before playing with Ruth. He now has 3 marbles. What has happened during the game?

- between compared quantities (no inclusion relationship)



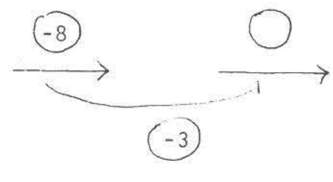
Example 5: Susan has 3 dollars in her pocket. Betty has 8. How much less does Susan have? Or how much more does Betty have?

- between transformations



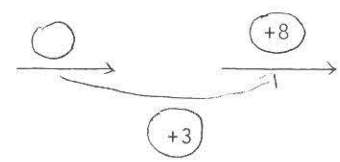
Example 6: Fred has played two games of marbles. In the second game he has won 3 marbles. He does not remember what happened in the first game. But when he counts his marbles he finds that he has won altogether 8 marbles. What happened in the first game?

Other examples could be:



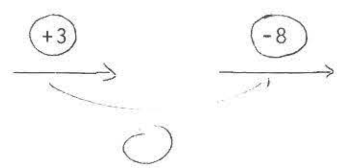
find the second transformation

or



find the first transformation

or



find the overall transformation

and so on.

One can easily imagine the difficulties that children may meet in extending the meaning of subtraction from their primitive conception of a "decrease" to all these different cases. Each case requires from the child some relational calculus (calculus of relationships) enabling him to choose the right arithmetic operation 8 – 3.

the study of such problem-solving situations [see Carpenter, Moser, Romberg, 1981]. The analysis of students' procedures and failures is most enlightening. It shows a slow development, over years and years, of children's conceptions of addition and subtraction. A psychogenetic approach is relevant and valuable not only for the elementary school level, but also for the secondary level. Think of the difficulties raised by "Chasles relationships" for 14-and 15-year-olds.

$$\overline{AB} = \text{abs}(B) - \text{abs}(A)$$
$$\overline{AC} = \overline{AB} + \overline{BC} \rightarrow \overline{AB} = \overline{AC} - \overline{BC}$$

There may even be a complete decalage between the student's conception and the mathematical model. For example, young children's conceptions of addition and subtraction fit better to a unary-operation model than to a binary-operation model. As a matter of fact, addition can be viewed either as an external and unary operation of Z upon N (in the discrete case), which models very well the increase of a quantity.

$$\boxed{q} \xrightarrow{(+t)} \boxed{}$$

$q \in \mathbf{N}$ $\quad (+t) \in \mathbf{Z}$ discrete case

$q \in \mathbf{R}_+$ (or $\mathbf{D}_+$) $\quad (+t) \in \mathbf{R}$ (or $\mathbf{D}$) continuous case

or as an internal binary composition of two elements of $\mathbf{N}$, which models very well the combination of two quantities.

$$\left.\begin{array}{c}\boxed{q}\\ \\ \\ \boxed{t}\end{array}\right\} \boxed{}$$

$q \in \mathbf{N}$ $\quad t \in \mathbf{N}$ discrete case

$q \in \mathbf{R}_+$ (or $\mathbf{D}_+$) $\quad t \in \mathbf{R}_+$ (or $\mathbf{D}_+$) continuous case

The two models are not totally equivalent to each other for students. This discrepancy has heavy consequences for symbolic representations of addition and subtraction, as we will see later.

Teachers cannot just ignore the fact that students' conceptions are shaped by situations in ordinary life and by their initial understanding of new relationships. They must deal with this fact and know more about it. It is an absolute necessity for them to know what primitive conceptions look like, what errors and misunderstandings may follow, how these conceptions may change into wider and more sophisticated ones, through which situations, which explanations, which steps.

It is essential for teachers to be aware that they cannot solve the problem of teaching by using mere definitions, however good they may be; students' conceptions can change only if they conflict with situations they fail to handle. So it is essential for teachers to envisage and master the set of situations likely to oblige and help students to accommodate their views and procedures to new relationships (inversion and composition of transformations for instance) and new types of data (large numbers or decimal numbers ..). This is the only way to make students analyse things more deeply and revise or widen their conceptions.

Solving problems is the source and criterion of operational knowledge. We must always keep this idea in mind and be able to offer students situations aiming at extending the meaning of a concept, and at testing student's competences and conceptions. This idea is crucial for researchers in France at the present time in our effort to provide a theory of didactic situations and operational knowledge.

Most obviously, this view leads to practical considerations in mathematics, and to practical goals of education. But I also want to stress that there is no opposition between practical and theoretical aspects of knowledge. They are both faces of the same coin, and one cannot think of many practical competences in mathematics that refer to no theoretical view whatsoever; competences are always related to conceptions, however weak these conceptions may be, or even wrong. I do not know of any algorithm or procedure that would develop and live by itself, free of any idea of the relationships involved. Reciprocally, theoretical concepts or theorems are void of meaning if they cannot be applied to any practical situation.

Still, a problem is not necessarily practical. It may also be theoretical; for instance, the extension of multiplication and division to negative numbers is mainly a theoretical problem: multiplying a negative by a negative does not refer to any practical problem (for 12-14 year-olds anyway), unless you consider the use of algebraic calculus as a practical problem. Theoretical questions may be related to different level competences. For instance, the conception of addition as a unary operation modeling the increase of a quantity, versus a binary operation modeling the combination of two quantities, concerns the practical problem of representing many real life situations; the extension of multiplication to directed numbers concerns the coherence of algebraic calculus. Both theoretical problems are related to competences, but not at the same level.

2. A developmental approach

Conceptions and competences develop over a long period of time. This is true not only for general characteristics of thinking such as studied by Piaget and other psychogeneticians, but also for the specific contents of knowledge. For example, the concepts of fraction and ratio have their roots in activities that are meaningful for 8-year-olds, for simple values as 1/2 or 1/4; and still the concept of rational number is a big and long-lasting source of difficulty for 15- or 16-year-olds and many adults.

As regards additive structures, although the first principles of addition and subtraction are understood by 3 or 4-year-olds, 75% of 15-year-olds still fail problems like the following ones:

Example 7: John has received 45 dollars from his grandmother. Then he goes to the store and buys different things. When he counts his money, he finds 37 dollars less than he had before receiving money from his grandmother. How much did he spend?

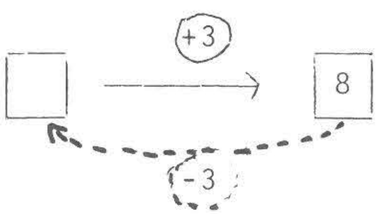
Example 8: Mr Dupont drives along the Loire valley 35 km westwards; then he drives eastwards. When he stops, he is 47 km eastwards from his departure point. How long was the second part of his drive?

The same 15-year-old students are taught Chasles' relationship $\overline{AC} = \overline{AB} + \overline{BC}$, which is directly related to example 8. But nothing is said, in the curriculum, about composition of functions which would help them in understanding these examples better.

Psychologists have described some general stages of intellectual development, but they have not paid enough attention to the detailed paths taken by students in developing specific competences like solving addition and subtraction problems of all kinds. A good example, though, is Noelting's work on ratio, but the problems envisaged are too limited, and Noelting's interpretation of the results tends to rely more on logical characteristics than on mathematical ones.

A priority for research in maths education and psychology is to make experiments on a large variety of problems so as to make a better picture of the steps by which students handle different classes of mathematical problems and use different procedures, not equally powerful, to solve them.

First of all, it is essential to recognize the variety of problem-structures and to analyze the relationships involved and the operations of thinking and procedures necessary to solve each class of problem. For instance it is not the same operation of thinking to invert a direct transformation as to find a complement (see examples 2 and 3 above).



To solve a find-the-initial-state problem like example 3 by inverting the direct transformation and applying it to the final state (dotted arrow), is not equivalent to making a hypothesis on the initial state (4 for instance), applying the direct transformation, seeing the difference from the expected final state and then correcting the hypothesis.

$$\boxed{4} \xrightarrow{+3} 7 \quad \text{(does not fit)}$$

$$\boxed{5} \xrightarrow{+3} 8 \quad \text{(OK)}$$

Obviously these procedures are not equally powerful and the second one has only a local value, depending heavily on the numerical characteristics of the variables involved. Small whole numbers have properties that help children to solve many problems by non-canonical procedures. A way to get students to move to more powerful procedures is to use larger numbers for which the canonical procedure is the only practical possibility. In some other problems it might be necessary to use decimal or rational numbers. Brousseau [1981] has insisted a lot upon the fact that changing numerical values is an important didactic means of making students' conceptions move from primitive to more sophisticated ones.

But of course this is not independent of the intellectual development of students; and the description of the comparative complexity of problems and procedures relies very strongly upon a developmental approach of mathematics learning. However, this description may be very partial and not even be understandable if the range of problems considered is too small and the period of development too short. For the same class of problems one may find a decalage of 3 or 4 years among students in the use of the same procedure, correct or wrong; and very often the behavior in one problem cannot be understood unless it is related to the behavior in other problems. There are strong correlations, strong hierarchies, and also lots of metaphoric substitutions in the handling of problems. This consideration has led me to the firm conviction that it is necessary to study the formation of rather large pieces of knowledge: conceptual fields.

But before I explain below what a "conceptual field" consists of, I need to stress my third point on invariants and theorems in action. I also need to give a short conclusion to this part on development.

The slowness of concept development is heavily underestimated by teachers, parents, and curriculae. For instance it is very often accepted that once students have studied a chapter of mathematics they should know it, or at least a high percentage of them should know it; and therefore it should not be necessary to come back to it during the following school years. Empirical studies show that it would be wiser to study the same field year after year, going deeper into the field each time, meeting new aspects, and coming again on aspects studied before. This should be widely done through problems to be solved. Different problems usually require the mastery of different properties of the same concept. It is essential to recognize this fact in a developmental approach.

3. Theorems in action and invariants

It is usually accepted, in education, that action and activity by students should be favored in order to push them to construct operational knowledge, i.e. lively and efficient knowledge. But it is not so usually recognized that action in situations and problem-solving is concept formation. In what sense is it concept formation? Here we come to a very important theoretical point.

Mathematicians know what invariants are and I do not have to explain in detail that, under certain sets of transformations (or variations), some quantities or some relationships remain invariant: actually invariants are a means of characterizing sets of transformations. But mathematicians and maths educators have not yet fully acknowledged the fact, which is more specific to cognitive developmental psychology, that very simple invariants, about which adults do not even think that they could possibly vary, are not invariants at all for young children, and are even most obviously variable, as first shown by Piaget:

- the number of eggs when you change the arrangement;
- the quantity of orange juice when you transfer it from a wide glass into a narrow one;

- the weight and volume of a piece of plasticine when you change its shape.

Isn't it obvious that there is more water in the narrow glass, since water gets to a higher level?

Invariants are a recurrent topic in Piaget's work, down to the problem of young babies' development and the "permanent object scheme". And yet Piaget has not fully recognized the importance of many invariants, most important in mathematics and physics: *relational invariants*. By "relational invariants" (this is an idiosyncratic expression) I mean relationships that remain the same over a certain set of transformations or a certain set of variations (a range of values for instance).

Let me give an example in parenthood relationships: it is not easy for a young boy to grasp the idea that the relationship "son of" is at the same time true for himself and his father, himself and his mother, his friend Matthew and Matthew's parents, and even his own father and his grandparents: how can his father be both father and son? Similar problems are raised by spatial relationships (behind, to the west of ...), numerical relationships (bigger than, bigger by n than, multiple of ...) and others.

Beside these binary relationships, children at an early age meet higher level relationships, which we usually call theorems. They do not meet them in a real mathematical shape of course; but they nevertheless do have to handle these theorems in action and in problem-solving, at least for certain values of the variables. This is my reason for calling them "*theorems-in-action*".

The essential purpose, for a cognitive analysis of tasks and behaviors, is to identify such theorems in action, even if it is not easy to do so and come to an agreement upon their behavioral criteria.

Let me start with the example of the third axiom of the theory of measure.

$$m\,(x * y) = m\,(x) + m\,(y)$$
$$\forall\ x, y \text{ for } * \text{ adequately chosen.}$$

In the case of discrete quantities and cardinals this axiom becomes:

$$\text{card}\,(X \cup Y) = \text{card}\,(X) + \text{card}\,(Y)$$
$$\forall\ X, Y \text{ provided } X \cap Y = \varnothing$$

Such an axiom is necessarily used by a young girl laying the table, when she counts the persons in the lounge and the persons in the garden, and adds both numbers to find out how many persons there are altogether.

It is a more efficient method than getting all the persons in the garden and counting them all.

A large amount of research work has been done on the initial learning of addition that shows the emergence of such a theorem in action.

The "counting-on" procedure, for instance, is a crucial step in the discovery of such a theorem (or axiom). It conveys another important theorem:

$$\forall\ m,n; \quad \underbrace{m \xrightarrow{+1} \xrightarrow{+1} \xrightarrow{+1} \cdots \xrightarrow{+1}}_{n \text{ times}} \quad \text{is equivalent to} \quad m + n$$

as an intermediary step between the counting-all procedure, which does not suppose the axiom, and the usual procedure in its final state, when $m + n$ is known as a fact or as a result of the addition algorithm [for more details, see Carpenter, Moser, Romberg, 1981, and especially Fuson's paper].

There are several other axioms and theorems involved in the construction of the natural number concept as the measure of discrete quantities. Some of them can be appropriated by 3, 4 or 5-year-olds. Some by 6 or 7-year-olds. Some are still difficult for many 9-year-old children.

o $2 \xrightarrow{+1} 3$ whatever objects are counted

$3 \xrightarrow{+1} 4$

...

o $2 \xrightarrow{+1} \xrightarrow{+1}$ is equivalent to $2 \xrightarrow{+2}$

...

$n \xrightarrow{+1} \xrightarrow{+1}$ is equivalent to $n \xrightarrow{+2}$

o $2 \xrightarrow{+1} \xrightarrow{+1} \cdots \xrightarrow{+1}$ is equivalent to $2 \xrightarrow{+n}$

...

$\underbrace{m \xrightarrow{+1} \xrightarrow{+1} \cdots \xrightarrow{+1}}_{n \text{ times}}$ is equivalent to $m \xrightarrow{+n}$

for $n + m \leqslant 10$

o conservation of discrete small quantities

o card $(A \cup B)$ = card (A) + card (B) provided $A \cap B \neq \varnothing$ and $A \cup B$ not too big.

o $X \subset Y \rightarrow$ card $(X) \leqslant$ card (Y) (inclusion problem)

Many other theorems can be identified that are necessary to the solution of addition and subtraction problems as illustrated above in examples 1 to 8. It would be boring to enumerate all of them; but I have said enough to conclude that the landscape is more complicated than one expects it to be at a first glance.

The study of multiplicative structures also shows the emergence of solutions to simple and multiple proportion problems that can be interpreted as the emergence of theorems in action: these solutions have not usually been taught to students and are not usually explicitly expressed by them. This is the case for the isomorphic properties of the linear function.

$$f(x + x') = f(x) + f(x')$$
$$f(\lambda x) = \lambda f(x)$$
$$f(\lambda x + \lambda' x') = \lambda f(x) + \lambda' f(x')$$

It would probably be nonsense to teach these theorems and procedures formally to students. It is better to face them with problems in which they may find it natural to use them (for simple values of the variables, for instance) and then help them to extend the procedure to other values of the variables.

Anyway these isomorphic properties are more easily used than the constant coefficient property.

$$f(x) = ax$$
$$x = (1/a)\,f(x)$$

even in cases when the numerical value of a is simple (3 or 4).

The psychological (and epistemological) reason for this, is that the variables x and $f(x)$ are not pure numbers for students, but magnitudes; and it is not easily accepted by students to look for the ratio of magnitudes of different kinds $f(x)/x$ or $x/f(x)$ (distance and time, costs and goods, weight and volume ...).

The above isomorphic properties of the linear function do not raise the same sort of difficulty because the extracted relationships relate magnitudes of the same kind [for further details, see Vergnaud, in press].

The isomorphic properties of the linear function are more easily used than the constant coefficient, even when the only procedure taught is the constant coefficient one. So we meet the paradox that a theorem in action that has never been taught as a theorem may be more naturally used than a theorem that has been taught but has not really become a real theorem in action.

This raises two questions:

- How can we make theorems become theorems in action?
- How can we make theorems in action become theorems?

Let us never forget that theorems in action are relational invariants. Like other invariants studied by psychologists, they are associated with a feeling of obviousness: they are at a certain stage of development taken as obvious properties of situations.

How can we make students catch as obvious the relevant properties of situations for simple values of the variables, and then generalize?

4. Conceptual fields

This is another idiosyncratic expression. Is it necessary? and what does it mean?

An interactive conception of concept formation considers a concept as a triplet (S, I, ζ).

S : set of situations that make the concept meaningful
I : set of invariants that constitute the concept
ζ : set of symbolic representations used to represent the concept, its properties and the situations it refers to.

I will deal with the symbolic representations in the next part of this paper.

Several considerations can be presented now:

First: a given situation does not involve all the properties of a concept. If you want to address all properties of a concept, you must necessarily refer to several (and even many) kinds of situations.

Second: a given situation does not usually involve just one concept; its analysis requires several concepts. For instance, additive structures require the concepts of measure, transformation, comparison, difference and inversion, the concepts of unary and binary operations, the concepts of natural and directed number, the concepts of function, of abscissa and others.

Third: the formation of a concept, especially when you look at it through problem-solving behaviour, covers a long period of time, with many interactions and many decalages. One may not be able to understand what a 15-year-old does if one does not know the primitive conceptions shaped in his mind when he was 8 or 9, or even 4 or 5, and the different steps by which these conceptions have been transformed into a mixture of definitions and interpretations. It is a fact that students try to make new situations and new concepts meaningful to themselves by applying and adapting their former conceptions.

As a consequence of these three reasons, I consider that psychologists and maths educators must not study too small-sized objects, because they would not understand the complex process by which children and adolescents master, or don't master, mathematics.

A "conceptual field" if "*a set of situations, the mastering of which requires a variety of concepts, procedures and symbolic representations tightly connected with one another.*"

This definition is not intended to be rigorous: it refers to a set of problems (not strictly defined) rather than to a set of concepts; the description of its contents requires both the analysis of situations and problems (i.e. the relationships involved) and the analysis of students' procedures when dealing with these situations. Symbolic representations such as diagrams, algebra, graphs, tables ... may be crucial for the extraction oj relevant relationships, but may also be misinterpreted by students and misleading. I will come to this point in the next part.

The best example of "conceptual field" I can give is the field of "additive structures" in which I have taken many examples. But there are many other examples.

- *Multiplicative structures*: understood as a set of problems requiring multiplications and divisions. Although not independent of additive structures they make a specific field that includes simple proportion problems (even simple multiplication and division problems are proportion problems) and multiple proportion problems which are (unfortunately) not often analyzed as such: area and volume, for instance, are rarely studied with the help of the n-linear function, which is actually essential for their understanding. Ratios of magnitudes of the same kind and ratios of magnitudes of different kinds lead to the concept of rational number and also to dimensional analysis (product and quotient of dimensions). Vector-space theory is also involved in the analysis of linear combinations and linear mappings.

Different sorts of symbolic representations may be useful to represent these problems: they are not equally meaningful to students. It depends on the problems and on the students' level of analysis: tables, graphs, equations have different properties.

• *Spatial measures*: such as length, area, volume make a specific field across additive and multiplicative structures, involving both geometrical representation of space and arithmetization of space.

• *Dynamics*: such as coordination of distance, time, speed, acceleration and force make a conceptual field of its own, very important for physics, but also important for mathematics in the development of such concepts as measure, function, dimension ... For instance, it has been shown by Laurence Viennot that the primitive conception of force, as proportional to speed, is still alive among university students and even, in tricky situations, among highly educated physicists.

• *Classes, classifications and boolean operations*: constitute another important conceptual field, related to other fields, but having its own specificity. It also develops from infants' first categorizations to the mastery of boolean operations and inclusion relationship.

5. Representation and the problem of adequacy between signifier and signified

Let me recall, in Table I, the main relationships involved in additive structures and the three criteria that account for the differences between the first three cases.

I have used distinct symbols to represent combination of measures, transformation of a measure, comparison of measures, and to represent natural numbers (measures) and directed numbers (transformations, comparisons and other static relationships such as debts and abscissas).

This choice facilitates communication. If I had used algebraic expressions some information would have been lost. All these relationships and problems can be represented by equations in **Z** (or **R**) but at the cost (and profit) of identifying different mathematical objects to one another.

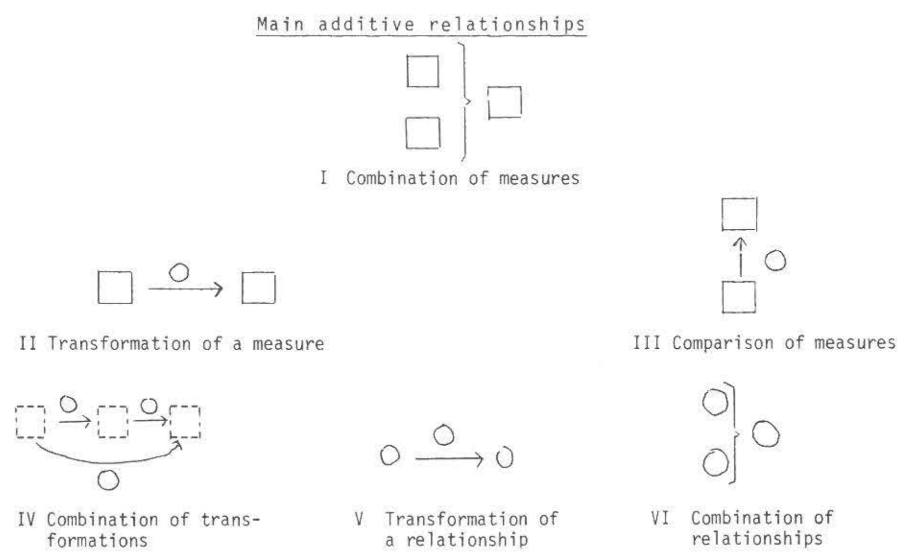
– identifying natural numbers to positive numbers (and decimal and real measures to $\mathbf{R}_{+}$.

– identifying the sum of measures, the application of a positive transformation, the combination of transformations, the combination of static relationships, and the inversion of a negative transformation to the same binary law of combination in **R** and its signifier "+".

– identifying the application of a negative transformation, the difference between measures, between states, or between transformations, and the inversion of a positive transformation to the same minus operation in R and its signifier "–".

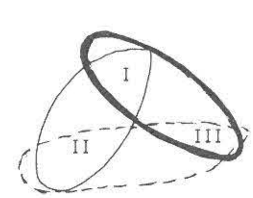
– identifying the equality sign with different meanings: is the same element as, outputs, is equivalent to.

Table 1



Criteria differentiating cases I, II and III

- presence/absence of a dynamic aspect
  II against I and III
- absence/presence of a directed relationship
  I against II and III
- absence/presence of a part-whole relationship (inclusion)
  III against I and II



This is all very good and necessary, but under which conditions, which explanations, and when?

The three criteria used in Table I to differentiate cases I, II and III show that case II is characterized by a dynamic aspect, the presence of a unary positive or negative operation and the presence of a part-whole relationship between the initial and final states. If case II is the primitive model of addition and subtraction for children, as most empirical results show, then one can expect some misunderstandings and some difficulties in the handling of other cases.

For instance, the equality sign may have for students a different meaning than for mathematicians, whose view is mainly shaped by case I (internal law of composition in $\mathbf{N}$).

By the way, Table I shows that, although distinct, three criteria do not make eight cases but only three: this comes from the fact that they are not independent. Still all three of them are useful tools in understanding the different difficulties met by students.

But let me give more examples:

Example 9: Peter has 5 marbles. He plays a game with friends and wins 7 marbles. How many marbles does he have now?

Example 10: Robert has just lost 7 marbles. He counts his marbles and finds 5. How many marbles did he have before playing?

Example 11: Thierry has just played two games of marbles. In the second game he lost 7 marbles. When he counts his marbles at the end, he finds that he has won 5 marbles altogether. What happened in the first game?

First, it is important to know that, although all three problems can be solved by the addition $5 + 7$, example 10 is solved about 1 or 2 years later than example 9, and example 11 is failed by 75% of 11-12 year-olds.

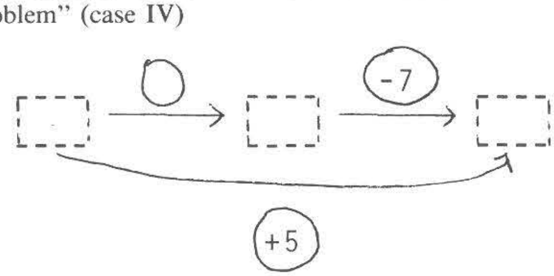
This can be explained by the fact that example 9 is a "find-the-final-state-problem" (case II)

$$\boxed{5} \xrightarrow{+7} \boxed{}$$

whereas example 10 is a "find-the-initial-state-problem" (case II).

$$\boxed{} \xrightarrow{-7} \boxed{5}$$

and example 11 is a "find-the-first-transformation-problem" (case IV)

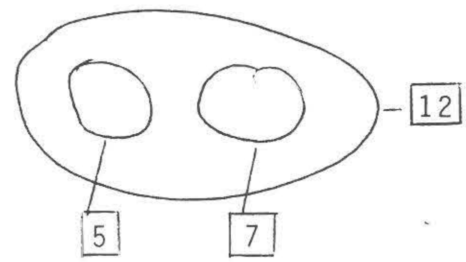


I will illustrate with three different symbolic systems (equations, arrow diagrams, Euler-Venn diagrams) some ambiguities of the solutions that can be offered by students:

Example 9 (see above)

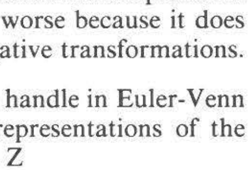
$$5 + 7 = 12$$

$$5 \xrightarrow{+7} 12$$



All three representations can be accepted and represent both the problem and the procedure to solve it.

Example 10 (see above)

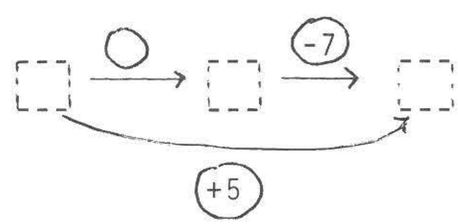
problem	procedure
$\boxed{}\ -7 = 5$	$5 + 7 = 12$
$\boxed{} \xrightarrow{-7} 5$	$5 \xrightarrow{+7} 12$
? ? ? No Euler-Venn representation of negative transformations	

The symbolic representations of the problem and the procedure are different. Most students, at the elementary school level, represent the procedure and not the problem. The Euler-Venn symbolism is still worse because it does not allow the representation of negative transformations.

Example 11 is still more difficult to handle in Euler-Venn symbolism, and the two correct representations of the problem are either the equation in Z

$x + (-7) = (+5)$

or the arrow diagram:



The solution that students usually write

$5 + 7 = 12$

(when they find it) has nothing to do with the representation of the problem.

Here is a last example:

Example 12: Janet has played marbles in the morning and in the afternoon. In the morning she won 14 marbles. In the afternoon she lost 31 marbles. When she counts her marbles at the end, she finds 23. How many marbles did she have before playing?

The arrow-diagram of the problem is the following

$$\boxed{x} \xrightarrow{+14} \square \xrightarrow{-31} \boxed{23}$$

Among the solutions, right or wrong, that I have found, I will cite four:

A. $23 + 31 = 54 - 14 = 40$
B. $23 + 31 = 54$
   $54 - 14 = 40$
C. $31 - 14 = 17 + 23 = 40$
D. $14 - 31 = 17$
   $17 + 23 = 40$

Solution A exhibits a treatment of the problem which is quite fair: starting from the final state, adding what has been lost and subtracting what has been gained. But the writing violates both the symmetry and transitivity of the equality sign.

Solution B does not violate any property of the equality sign. It may be considered to be better. But it is essentially the same procedure as A. Like A it represents the procedure and not the problem. The equality sign is probably taken as an output symbol, not as an equality relationship.

Solution C leads to another comment. Although it also violates the symmetry and transitivity of the equality sign, this procedure consists of steps which are different from those used in procedures A and B: it combines transformations first, and then applies the result of the combination to the final state.

Solution D, which is essentially the same as C, shows a new mistake, $14 - 31 = 17$. But if you think of it, it is not a mistake in the context of the particular procedure: the problem is to find the difference between two transformations, one positive (+ 14) and one negative (- 31). Is the student right to be happy with his solution? My answer is yes.

So one can see the many sources of decalages between signifiers and signified in additive structures and problems.

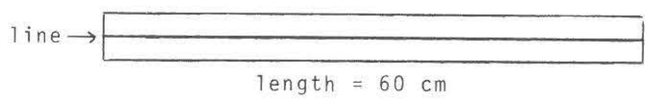
Some symbolic systems are quite unable to represent problems that imply certain relationships. Some symbolic systems are not likely to help students to distinguish between the representation of problems and the representation of solutions. And finally some symbolic systems may convey meanings that stand a long way from mathematical standards. One big problem for research on teaching is "how can we fill the gap, or help students to fill it?

I will now turn to a different example, which is clearly related to additive structures, although it is more important in geometry and graphs: the real number line.

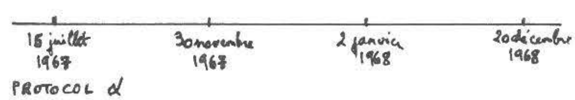
The real number line considers numbers as dots on a line, positive on the right hand side and negative on the left hand side of an origin called 0. This is clearly a symbolic system. What operations of thinking does it require from students to read it and use it?

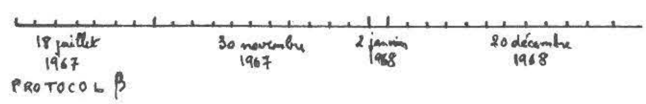
We have carried out experiments on this in Paris. The report has not yet been published and only a short description has appeared [Vergnaud and Errecalde, 1980]. I will only mention the main aspects of our results.

Having to place numerical data (weights, distances) or quasi-numerical data (ages, dates of birth) on a long strip of paper, with a line in the middle students from 10 to 13 meet many kinds of difficulties. We collected more than 600 protocols, and not less than a dozen criteria led to 50 or 60 different categories.



Let me start with a few questions about two protocols concerning dates of birth.

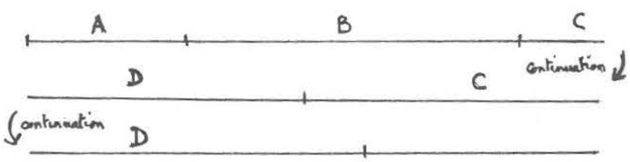




The author of protocol α is satisfied with ordering regularly dots representing the dates, taking no account of the different durations between dates. The author of protocol β represents dates as segments, the length of each segment corresponding to the rank of the month in the year. Segments are placed end to end (no inclusion relationship between durations). The day and the year are ignored. There is a confusion between event and duration and between the ordinal and cardinal aspects of the data.

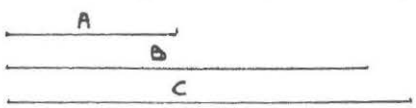
Which protocol is nearer the final concept of scale or number line? It would be hard to say because neither of them is close to the final concept, which requires a synthesis between the concept of order and those of distance and interval. But I consider protocol β as a very first attempt to take durations into account. There would not be much to say about all this, if protocols α and β were just anecdotal exceptions. But nearly one third of 10-year-olds' protocols are in categories similar to α and β .

A very important obstacle for students at that level consists in the inclusion principle. The end-to-end protocols illustrate the principle: "distinct signifiers for distinct signifieds", or else "no inclusion between excluded quantities". When they represent weights a, b, c, d ... of newly-born babies, many students produce such protocols as the following:

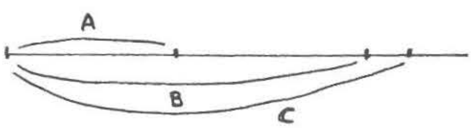


or they may cut the data and place end-to-end only the decimal parts of them:

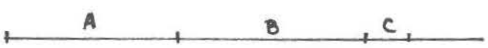
To overcome this difficulty, there are several steps, either



or

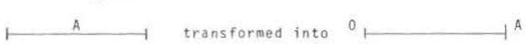


or

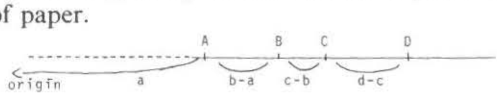


The last protocol is ambiguous because B is just above the segment that represents the difference $b - a$ (and not b). The same applies to C.

At this stage there are still some steps to go. One of these steps consists in the identification of segments with the right hand extremity, the left hand extremity being identified as the origin.

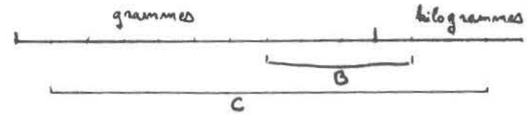


Another step consists in reasoning on dots and intervals whatever the origin may be, even if it is not present on the strip of paper.

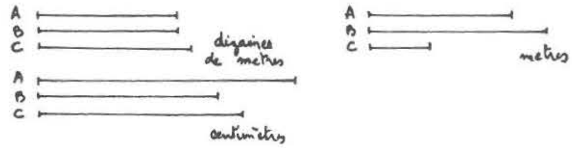


In two of the tasks used, the choice of the data and the scale did not make it possible to place both the origin and the data on the same line. So students were obliged to change origin and/or reason on differences ($b-a$, $c-b$, $d-c$ ...). Most of them were unable to do so, even after five or six lessons on scales.

Other protocols show that the coordination between two systems of units (meters and centimeters, kilogrammes and grammes, years and months) also raises big difficulties. Some protocols represent the same data by two dots on two separate scales.



or even three segments in three different systems of units



or even two dots on the same scale (reading it with one unit above and the other unit beneath).

In summary, reading and using a symbolic system like a scale, directly related to the very important mathematical concept of "number line", requires several operations of thinking that children and young adolescents do not find natural at all.

Whereas it is fairly natural for children to represent ordered magnitudes by ordered dots on a line, or magnitudes by separate and distinct segments, it is most difficult for them to coordinate both points of view and to accept the inclusion principle: $0A$ not only represents a, but also parts of all other data satisfying

$$a < b < c \rightarrow 0A \subset 0B \subset 0C$$

even when 0 does not appear on the line.

6. Problems of methodology

Methodology is usually important in the emergence of a new scientific field. This is also the case for research in maths education. Psychology provides us with some sufficient tools, for example clinical and critical interview techniques, designed experiments and questionnaires.

It is fair and very important to use these methods as often as possible. They can provide reliable and analysable information. But they probably miss an essential point: the description and analysis of the process taking place in the classroom, where interaction with new situations and new objects on the one hand, interaction with the teacher and the other students on the other hand, make the picture more complex and more evolutionary.

This paper is already very long, and I don't want to repeat the well-known advantages and defects of interviews, questionnaires and designed experiments. All of them are useful techniques. Each of them can be used, according to the particular question raised. Interviews are better for the understanding of students' conceptions, designed experiments for reliable comparisons and for control of the comparative importance of different factors, questionnaires for large-scale assessments.

I would rather insist upon the most specific method of didactics: experimenting in the classroom. This is not an easy job. It requires long preparation, a good-sized and well-trained group of teachers and observers and costly registration devices.

A time-consuming part of the work consists in reading the recorded tapes.

What is most important, still, is the necessity to take steps to get reliable information. Many observers in the classroom only see anecdotes, the importance of which can hardly be estimated. The challenge is to promote didactic experimentation in the classroom as a reliable method providing repeatable facts.

I would like to stress three important needs:

1. the need to make as explicit as possible the cognitive objectives of the sequence of lessons undertaken;

2. the need to review carefully beforehand the choice of the situations, the reason for sequencing them so and so, the modalities in which information is given and the question asked, the numerical values of the variables, the symbolism used and the explanations that should be provided;

3. the need to make explicit hypotheses about the behavior of students and the events that might happen.

One cannot observe well what one is not prepared to observe. This presupposes that the contents, and the situations through which these contents are conveyed, are clearly analyzed beforehand so that one may be prepared to "see" the meaning of events and behaviors observed. It would be unrealistic to expect to reach a high degree of reliability and repeatability immediately. Psychologists were faced with the same problem when they started experimenting and interviewing: they were also faced with the extreme variety of subjects' behaviour. But when interviewing subjects you notice that regularities appear, and the more subjects you see, the more stable the different patterns of behavior appear to be.

Researchers are not in a position to give many examples of repeated experiments in didactics. But it is a fact that, when sequencing and observing the same series of lessons in different classrooms, with the same teacher or with different teachers, at different levels or at the same level, one can observe that some events happen and happen again and the same coherently and hierarchically organized behaviors appear and appear again.,

Errors, procedures, spontaneous explanations and formulations, ways of designating and representing things are most enlightening.

The specific and rewarding aspect of the method is the observation of the process of discovery by students, of the contradiction between their initial conception and the situation they have to handle, of the conflict between different students' points of view (working in groups of three or four), of the evolution of conceptions and procedures. I can't see how we will be able to develop a scientific approach to maths education without experimenting in the classroom, under conditions that make this experimentation as scientific as possible.

The cost is heavy, and many improvements have still to be found. But when you have to pay the price the only thing to do is to pay the price. Experimenting in the classroom is an inescapable issue.

Conclusion

My conclusions have already been given all through this paper. I will not repeat them again. I can only insist once more upon the analysis of the mathematical contents involved in situations. This analysis is essential. I also want to stress the importance of a behavioral and developmental approach to learning, and the importance of understanding students' actual conceptions, behind their behaviors and verbal explanations. I finally think that one must try to avoid all kinds of schematism, because the landscape is complicated, and because meaning and understanding cannot be handled without taking into account metaphors, misunderstandings and the strange relationship between words and meanings, between signifiers and signified.

I have sometimes been told by North American friends that they were surprised by the "struggle for theory" in France. We may "struggle" too much. But some examples I have given are convincing examples and show that symbolic representations are really a crossroads in maths education research.

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