Communications

'The likeness of unlike things': insight, enlightenment and the metaphoric way

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> Mathematical beauty is the expression mathematicians have invented in order to obliquely admit the phenomenon of enlightenment while avoiding acknowledgement of the fuzziness of this phenomenon. […] This copout is one step in a cherished activity of mathematicians, that of building a perfect world immune to the messiness of the ordinary world, a world where what we think *should* be true turns out to *be* true, a world that is free from the disappointments, the ambiguities, the failures of that other world in which we live. (Rota, 1997, pp. 132–133; *italics in original*)

There are two other articles which pertain to metaphor in this issue of FLM – by Zwicky and by Font and colleagues. I intend to look at aspects of them both and end by making reference to a new book by Reviel Netz (2009) on Hellenistic mathematics (centred on, but by no means limited to, the work of Archimedes), while attempting to tease out further aspects of metaphor in relation to mathematics along the way.

Since Lakoff and Núñez (2000) produced their 'conceptual metaphor' account of mathematics' origins, in mathematics education at least there has been much talk once again of metaphor in relation to mathematics. Their work underscored the cognitive dimension of metaphors in the development of mathematics, but did so without talking about the actual conscious production of metaphors by mathematicians (see Schiralli and Sinclair, 2003). Earlier too (see, for instance, work by Sfard, Presmeg or myself), metaphor was not felt to be totally out of place in discussions of both mathematics and mathematics education (not least due to the considerable use of semi-sanctioned pedagogic metaphors in the school teaching of mathematics). However, it has always been a far from uncontentious area, especially in relation to the philosophy of mathematics. [1]

In her essay 'Mathematical analogy and metaphorical insight', reprinted here, philosopher-poet Jan Zwicky carefully argues for a strong parallel between mathematical analogies (and the insight they bring) on the one hand and metaphors (and the insight they bring) on the other. In her reply to a rejoinder to her original piece in the *Mathematical Intelligencer,* Zwicky (2007) adds:

> To many of my colleagues in the humanities and to several in the sciences (*vide* Pólya quoted in my essay), it is anything but obvious that metaphorical contemplation constitutes a way of thinking. […] It is because of this widespread denial that metaphorical thought is genuine thinking *aimed at truth* that I felt it useful to develop the correspondences with mathematical insight. (pp. 13–14)

It is interesting to me that Zwicky seems at pains in her piece not to use the expression 'mathematical metaphors' (completely understandable given the original audience for which she was writing as a mathematical outsider, albeit a very far from uninformed one). Yet in so doing, she illustrates a metaphorical way of working in her article itself. It is as if she is bringing two objects close to one another to examine their degree of attraction and repulsion. But one is more familiar than the other: this one serves as touchstone at some times and as lodestone at others.

In addition, she offers this specific comparison as both a way to argue for metaphoric thought *and,* tacitly to my eyes, provides a way to help us think about mathematical analogies. The converse is more commonly the case, namely mathematics being used as a putative source of illumination of the non-mathematical. With regard to the very word 'analogy', Szabó (1978) is insistent that:

> Greek grammarians have used this word [αναλογία] with its present day meaning since Hellenistic times.
>
> It is less well known that this same word was not originally a grammatical or linguistic term, but a mathematical one […] The Greek grammarians of Hellenistic times undoubtedly borrowed their term αναλογία from the language of mathematics. So in the last analysis we are indebted to Greek mathematics for our word 'analogy'. (pp. 145–146)

This is not a lone example of a term's mathematical use predating its wider acceptance as conceptual currency in different realms. David Fowler (1987) asserts that the notion of palindrome was likely a phenomenon first noticed and named by mathematicians in working with anthyphairetic sequences in their exploration of certain 'irrational' ratios.

But these instances do tend to reinforce the presumption of the mathematician looking out at the world through mathematical eyes, whereas I read Zwicky as inviting us to explore the opposite: poets work extensively and powerfully (though certainly not exclusively) through metaphoric thinking. How might this expertise be deployed towards understanding aspects of mathematics better?

I am caught by her partially implicit claim on the first page that metaphors can serve to *align* the world. It speaks of making the world not in but rather according to your own image. Such agency is involved, such deliberation, such power. Metaphors are indeed made: they also partake, perhaps, of those rare instances where saying something makes it so (albeit tacitly tentatively and initially temporarily). [2] Once 'institutionalised' by a variety of means, however, they become part of the resources of the particular language (and possibly translated across languages), sanctioned ways of seeing. [3]

To make a metaphor is to make an assertion: it has epistemic content. For instance, at a meta-level, to assert that a poem is a deer or a poem is a cloud chamber (as I have done in poems), or that a theorem is a calm, an equanimity or, in Mulcaster's gorgeous phrase, 'a likeness of unlike things' (see Pimm and Sinclair, 2009a), is to make a claim about the way some aspect of the world *is.* And *is*-statements are central to mathematics: definitions, theorems and metaphors all share this same linguistic form. Therein, I believe, lies a central cause of unease.

If I assert a poem is a deer and I also assert a poem is a cloud chamber (as I have done), am I necessarily thereby asserting a deer is a cloud chamber. If I see the two initial 'is'es as equivalents, then apparently yes I am. However, along with Hamlet, and wind southerly or no, I know a deer from cloud chamber. I am thoroughly with Zwicky in seeing a poem as an intellectual investigation, embedded in a certain sort of rhetoric as an aid to eliciting conviction and offering an understanding. There is also an allusive quality often, that involves speaking the general both through specifics and what is laughingly called the concrete (recall Hadamard's observation, "The concrete is the abstract made familiar by time"). Metaphor relies on the imaginative power of humans to transform the world.

Yet there is a partial paradox: metaphors, at times, seem conservative; they frame the new in terms of the old, the already-known, the better-worked-out, the more-familiar. In so doing, they can miss, as McLuhan often observed, the essence and sometimes the shock of the genuinely new. It is curious because metaphors are frequently novel in terms of the ways of seeing offered; but the components thereof are, of necessity, familiar. Metaphors, though backward-looking, are aimed at the future. But how quickly they can seem to vanish into the underbrush when they have done their work (see Pimm, 1988).

A second sentence from Zwicky's article, concerned with the question of *necessity and truth,* really caught my attention:

> mathematics shows us necessary truths unconstrained by time's gravity; poetry, on the other hand, articulates the necessary truths of mortality. (p. 13)

This evoked for me the opening quotation by Rota, in terms of the distinction between two 'worlds', with mathematics firmly located in the 'upper', disembodied realm. With regard to mortality, it certainly seems true that mathematicians try to bury their metaphors, along with mathematics' objects, so both can resist disinterment (a necessary task of a mathematics teacher perhaps).

Catherine Chevalley, writing of her father Claude, a core member of the Bourbaki group, observed:

> For him, mathematical rigour consisted of producing a new object which could then become immutable. If you look at the way my father worked, it seems that it was this which counted more than anything, this production of an object which, subsequently, became inert, in short dead. It could no longer be altered or transformed. This was, however, without a single negative connotation. Yet it should be said that my father was probably the only member of Bourbaki who saw mathematics as a means of putting objects to death for aesthetic reasons. (in Chouchan, 1995, pp. 37–38; *my translation*)

So mortality may be an interesting phenomenon, not just for the mathematician (seeking immortality through mathematics), but for the dreamed mathematical objects too.

Mathematics claims to be about essences, about the way its objects are: this ontological desire is that of the mathematician. And Zwicky (2003), in her book *Wisdom and Metaphor,* asserts: "Ontological attention is a form of love" (p. 57L). [5]

The work of Font and Godino (also in this issue) draws on that of Lakoff and Núñez. They write about what they term 'the object metaphor' (mathematical objects ARE physical objects), noting it as being an especially problematic for teachers and students of mathematics. They refer to it as an 'ontological' metaphor, a means for seeing how things are, as well as a way of speaking about them 'as if'. (In Pimm, 1987, I talk about 'the symbol is the object' as a potent metaphor related to mathematical practice and the speaking about practice that teachers must engage in.). And their attention to synecdoche, another linguistic trope [4]

Font *et al.* also draw attention to the fact that the verb 'exist' as is commonly used in mathematics can have some significant issues, especially to my eyes in the negated form of 'does not exist'. It is here that the challenge arises of being able to name things which 'do not exist'. Non-existence in the material world can have a temporal element: a stegosaurus used to exist but now no longer does. A unicorn has never existed there (though the name does). But what about mathematics and its objects? What possibilities with regard to existence are carried by the object metaphor, as well as the fact that we can (and do) name non-existent objects? [6]

Many mathematicians over time have had difficulty with movement and traces of the temporal in mathematics, with 'time's gravity' (see Pimm, 2008). Are mathematical objects brought into being (only to be put to death again, so they can no longer be harmed by this world of change)? Are metaphors their midwives, who then subsequently take their leave, thereby permitting the claim not only of 'is' but also 'have always been'?

The mathematician William Thurston (1994) speaks of the importance of intuition, association and metaphor in his account of his own mathematical research (*e.g.,* "the Godbillon-Vey invariant measures the helical wobble of a foliation", p. 173) as well as "unexpected kinships" across mathematical fields. But right at the outset, he asserts what mathematicians are doing is "finding ways for *people* to understand and think about mathematics" (p. 162; *italics in original*).

Poets, likewise, I think Zwicky would contend, are involved in finding ways for people to think about and understand *almost anything* and metaphor is one central means they draw on. My reason for the subtitle of this piece including the phrase 'the metaphoric way' is that poets and mathematicians, to different degrees and in somewhat different circumstances though not for different purposes, both draw on this hypothetical way of thinking and bringing forth the world.

Mathematician André Weil (another core member of the Bourbaki group and brother of Simone), in speaking about human motivation in mathematics, writes:

> nothing gives more pleasure to the researcher [than] these obscure analogies, these murky reflections of one theory in another, these furtive caresses, these inexplicable tiffs. (1992, p. 52)

The metaphoric way in mathematics: What Is a triangle?

> Perhaps I have already said somewhere that mathematics is the art of giving the same name to different things. (Poincaré, 1913/1946, p. 375)

To take another look at a mathematical example that I have discussed elsewhere (Pimm, 1987), I start with the question, "why is the usage 'spherical triangle' metaphoric?".

The triangle is a significant configuration in the study of plane geometry. There are many theorems involving phenomena related to triangles: from similarity and congruence conditions to Pythagoras's theorem to an area formula in terms of the semi-perimeter to Napoleon's theorem, asserting that from *any* starting triangle, after constructing an equilateral triangle on each side, if the centres of each equilateral triangle are subsequently joined, that final triangle must also always be equilateral (see Davis, 1997, on his attempts to prove this result).

'Spherical triangle' labels a configuration on the sphere (roughly, three distinct points joined pairwise by arcs of great circles) and carries with its name the implicit suggestion that it plays for the sphere a similar role to the one that 'triangle' does in the plane (for otherwise why would it be called 'triangle' and not something completely other?). And much mathematics can be undertaken exploring which results 'carry over' to the sphere and which ones do not. One of the surprising ones is that the notions of similarity and congruence coalesce on the sphere, in that if two spherical triangles are similar then they must be congruent.

'Similar' and 'same' are central ideas in mathematics that echo strongly, for me, the notions of simile and metaphor. They both are positive, in that they emphasise sameness over difference, while both in their own ways contain a certain reticence in their identification, holding back from complete absorption or assimilation. Knowing the difference between the ways 'triangles' behave in the plane and on the sphere acts as a reminder that, in some situations, I may want to distinguish them and in others I may not. For me, there is something very strong about the metaphoric assertion of 'is' (one that brings it fascinatingly close to the usually un-remarked literal). The poem itself frequently explores a sense in which 'this' is 'that': a theorem likewise. This may be an instance where there is a two-way flow of insight and understanding.

I can go further: there is also a hyperbolic triangle. In the history of geometry, there is an interesting difference about hyperbolic geometry, which is that the geometry (in the sense of the theorems) occurred before the discovery/invention of (various) space(s) of which it was the geometry (the telling of the place preceded the place itself). This has led to the question relevant here of the difference between models of hyperbolic geometry and hyperbolic geometry itself. How indeed can we know the dancer from the dance? And students regularly ask if this one the real one (the 'real thing' that is love's true ontological object?).

Ancient affinities for metaphor: playful mathematics and playful poetry

In his most recent book, *Ludic Proof: Greek Mathematics and the Alexandrian Aesthetic,* Reviel Netz (2009) draws fascinating links, among other things, between mathematics and poetry in Alexandria during the Hellenistic period, in a focused exploration of what might be meant by mathematical style. There is no space here to go into much detail, other than to note that Netz draws clear and significant parallels between the two activities and, specifically, the forms of writing employed.

In one chapter, entitled *Hybrids and Mosaics,* Netz explores aspects of scientific (especially mathematical) nomenclature in relation to the introduction of metaphoric naming, linking it to a shift in written mathematical style and voice (see, too, Pimm and Sinclair, 2009b). One example he explores is that of Eratosthenes' "sieve" (an image that has persisted for more than two millennia), contrasting it with the apparently neutral literality of 'dodecahedron' (the 'twelve-faced'):

> within the context of the precisely literal and economic Greek mathematical discourse, the presence of metaphor would be an extreme example of the presence of authorial voice. To call this operation by the name "sieve" would not arise naturally from the impersonal mathematical operation itself: it would be an authorial statement, vividly presenting to us the thought of Eratosthenes itself.
>
> More than this: it would present to us a juxtaposition of the world of mathematical objects with that of a concrete, in this case quite mundane[,] activity. (p. 150)

He documents a range of mathematical examples of metaphoric naming, which he categorises thus:

> the humble (sieve, shell, leather-knife, salt-cellar), the rejected and lost (lock of hair), the suspect even: the ivy, above all, was a symbol of drunkenness. The scientific game of Hellenistic geometry is no place for sobriety: it is based on the jarring juxtaposition of the literal and the metaphorical, and specifically on the jarring juxtaposition of the abstract world of science with a humble stratum of human life. [...] The themes of Alexandrian naming, then, are metaphor and bathos (pp. 156, 159)

Individual metaphors have their histories, as does the phenomenon of metaphoric naming within mathematics, as Netz's intellectual archaeology reveals. His list evokes a possibly deliberate crossing of the boundary realms alluded to in Zwicky's piece and evoked in Rota's comment quoted at the outset: *memento mori.*

There is so much more that could be said. I am interested in pursuing a metaphoric way that is both productive and hypothetical; more interested in the parallels between mathematics and poetry rather than emphasizing their dissimilarities; interested in the sense of truth and necessity that involves time's gravity and the ways in which mathematics (and mathematicians) attempt(s) to shrug off this influence; interested in the power of names and naming in framing our experience of the world in all its forms.

Notes

[1] Metaphor has always had more play in science. Readers of this journal are likely familiar with philosopher of science Mary Hesse's (1966) work on models, metaphors and analogies in science, which reaches back at least as

far back to Aristotle's account of analogy as analogical to mathematical proportion (*e.g.,* in his *Topics*). Hesse talks of positive, negative and neutral aspects of an analogy (referring respectively to already known similarities, already-known dissimilarities and open areas for exploration), as well as arguing for their core role (including metaphor) in the *doing* of science. The neutral analogy, naturally, is the most potentially hopeful and hypothetical.

[2] The Greek verb *poein,* the root of both 'poem' and 'poetry', means "to make or compose". Thus, poems, like metaphors, are made.

[3] Not everyone is enamoured with certain as-if ways of speaking in mathematics, not least if it seems to signal motion or action. Plato, for instance:

> Their [geometers'] language is most ludicrous, though they cannot help it, for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of squaring and adding and applying and the like, whereas in fact the real object of the entire study is pure knowledge. (in Molland, 1991, p. 182)

However, Gilles Châtelet (2000), in his work *Figuring Space,* draws attention to the significance of gesture in mathematics and the core role of diagrams as intermediaries between the human and the mathematical. For him, metaphor and gesture are the two ways of transducing the mobility of the body into symbol.

> A diagram can transfix a gesture [...] capture gestures mid-flight; for those capable of attention, they are the moments when being is glimpsed smiling. Diagrams are in a degree the accomplices of poetic metaphor. (p. 10)

[4] In Zwicky's paper, she points at a trope within a trope, when speaking of 'good' and 'bad' metaphors, and ends with the suggestion that our sense of this is potentially a form of feedback from the world.

[5] This book is a remarkable achievement that deserves to be much widely read within mathematics education. Facing pages, number left and right, pair and juxtapose quotations of others, usually about poetry or mathematics (placed on the right) with observations and remarks of her own (positioned on the left). Thus each double-page spread is both a bringing together of texts and a visual embodying of metaphor.

[6] The final part of their paper, involving synecdoche (the classical instance of metonymy), relates to the questions of particulars in mathematics and echoes a common practice in much contemporary poetry of eschewing explicit generalities almost completely, in favour of working through particulars. Mathematics educators have much to learn from poets about how to do this. See Pimm and Sinclair (2009a).

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What is Ten? Relationships between language and numeration

CHRISTOPHER DANIELSON

How should we read the following number aloud?

$$200_{\text{four}}$$

Before reading further, read this number aloud and write out in words what you have said. Then consider what other possibilities exist.

I teach mathematics content courses for intending elementary teachers early in their studies (primarily first and second year college students). The topic of place value is *de rigueur* in textbooks for such courses and the study of the structure of alternate bases and of algorithms in alternate bases is a common (although not universal) approach. Such approaches rarely address the language of place value. When they do address language (*e.g.,* Sowder, Sowder & Nickerson, 2008), students are generally told to read numbers in alternate bases digit-by-digit. This is reasonable when alternate bases are being studied in the abstract. In courses like those I teach, however, alternate bases have a context; we teach them in order to inform our future teachers' understanding of the mathematics they will someday teach. Number language is an important part of that mathematics.

The remainder of this communication consists of an extended argument that the relationship between language and numeration is (1) problematic and (2) deserves careful attention in the preparation of elementary teachers. I make this argument by stating five possibilities [1] for reading 200_{four} aloud and commenting on each. The five possibilities are, in order of appearance in the article:

1. Thirty-two
2. Two-zero-zero base four
3. Two hundred base four
4. Two sixteens, and

5. Two blorks.

Thirty-two

It is mathematically correct to write:

$$200_{four} = 32_{ten}$$

That is 200_{four} is in fact equal to thirty-two. But to read 200^{four} as thirty-two suggests that English number language is independent of numeration, which is false.

This is a problematic suggestion to give English speaking future elementary teachers. Indeed their relationships between numeration and language are a focus of study in elementary school mathematics. If we suggest-even implicitly-to our future teachers that there is no relationship here, we miss an opportunity to fully prepare them for their future work.

Consider an example. The IMAP project (Philipp, Cabral & Schappelle, 2005) includes a videotaped interview of a student, Talecia, who is asked to solve 638 + 436 using left-to-right algorithm that has recently been demonstrated for her. Talecia says aloud, *Six hundred plus four hundred is ten hundred* and she writes *110*. The interviewers expend substantial effort in helping her to write correct symbols for her correct, though unconventional, language.

It has been suggested that preservice elementary teachers see mathematics as a straightforward, predetermined set of procedures for obtaining correct answers. [2] Seen in this way, Talecia's answer is wrong. *Ten hundred* is not written *110*. A more nuanced view is important for helping students like Talecia to make sense of the decimal system. If we suggest to future teachers that there is no connection between English number language and decimal numeration then we reinforce their tendency to see Talecia's response as nonsense rather than as a struggle to connect numeration and language.

Two-zero-zero base four

When textbooks for preservice elementary teachers pay any attention to the relationship between language and numeration, they generally use this possibility (*e.g.,* Sowder, Sowder & Nickerson, 2008). Unlike the previous possibility, which suggested that there is no relationship between numeration and language, this response suggests that there is such a relationship, but that its precise nature is not important to study. While this is never stated, my assumption that proponents of this approach do see a relationship between language and numeration is evidenced by the new linguistic construction that is introduced to read numbers in this alternate numeration system; few people would argue for reading:

32

as *three-two* or even *three-two base ten,* even if they would argue that:

$$200_{four}$$

should be read as *two-zero-zero base four.* In short, this approach introduces new number language for a new number system and this must be because language matters.

But this new digit-based language supports a troublesome conception of place value systems, the *concatenated digits* conception (Fuson *et al.,* 1997). In this conception, children see the number 32 as two juxtaposed digits rather than a way of writing *3 groups of 10 plus 2.* But the concatenated-digits conception is not just a problem for young children. Thanheiser (2009) found this conception among preservice elementary teachers prior to taking a content course. Preservice teachers with this conception of the decimal system were unable to explain the meaning of the *borrowing* or *regrouping* steps in the standard American algorithms for addition and subtraction of multi-digit numbers. They were unable to think of the digit *1* as *1 group of ten* and their explanations of regrouping involved writing digits without reference to place value.

In short, reading 200_{four} as *two-zero-zero base four* potentially reinforces incorrect conceptions of the decimal system held by intending elementary teachers. There are alternatives that have the potential to help preservice teachers to confront these conceptions and develop new, richer conceptions of place value that may directly and positively impact their teaching.

It is worth noting that at least one language does in fact read 32 as *three-two.* In American Sign Language (ASL), most two-digit numbers are signed in the format:

first digit-second digit

The number 100 is signed *one-hundred* (not *one-zero-zero*), so ASL number language has some relationship to place value, but this relationship is more complex than that of English.

Two hundred base four

One assumption behind this response is the same as the one behind the question in the title of this paper: ten means one group not this many things:

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Another assumption, perhaps coincident with the first, is that *ten* is a set of symbols: 10. Therefore, *one hundred* is also a set of symbols: 100.

So what is *ten*? Is ten a quantity? Is it the juxtaposition of the digits 1 and 0? Or is it a single group of the relevant base? If *ten* always refers to the number of asterisks above – that is, if the word *ten* is inextricably linked to a quantity, then similarly *one-hundred* refers to ten groups of ten and so *two-hundred base four* is a nonsense construction. But if *ten* is a group (whose size can vary), then *one-hundred* is also and 200_{four} is *two-hundred* as it represents two groups of (four groups of four). Similarly if *ten* is a set of symbols – a 1 followed by a 0, then 200_{four} is read as *two-hundred* because *one-hundred* must also be a set of symbols – a 1 followed by two zeroes.

Two sixteens

The correspondence between English number language and the decimal system is imperfect. [3] It has been noted in other places (*e.g.,* 1}Fuson, 1990) that many Asian languages match the decimal system perfectly. In Japanese, *sixteen* translates as *ten-six* while *thirty-two* translates as *three-tens-two.* The English word *sixteen* has less correspondence to the symbols 16. Indeed, it is plausible that many children learn the word as the next term in an arbitrary sequence rather than as *ten plus six.* If we think of *sixteen* as a word for this many objects: ****************, then 10_{four} could be read as *four,* 100_{four} as *sixteen* and 200_{four} as *two-sixteens.*

Two blorks

If we reject *sixteen* as not being far enough removed from the decimal system (the French *seize* is even further removed, being a single syllable containing neither the words *six*-6-nor *dix*-10), then we can develop a new word for 100four. Perhaps 10_{four} could be read as *flub*. Then there are *flub flubs* in 100_{four} and this could be read as *blork* (or *one blork*). Fuson (1990) has referred to such a system as a *regular named-value* system, whereas English is an irregular named-value system due to having a base-10 structure but numerous irregularities in the correspondence between decimal numerals and number words *(e.g., sixteen*).

Richardson (1998) has developed an approach for use with elementary students that involves work with alternate bases and creating new words for the groups. Second grade students play a game in which they count blocks, grouping in fours, then grouping the groups of four in fours. Students create a nonsense word to name the group (*e.g., flub*). New versions of the game are played with groups of 5, then of 6 and finally of 10. As they play each version, students create a different word for the corresponding group. Thus, once *flub* has been chosen to represent groups of four, all work in base four is referred to as the *flub game* and a new word (*e.g., flop*) is needed for groups of five. Ultimately, the class builds to groups of ten and so ten is a group of a particular size, rather than a set of symbols.

I have lately used an instructional approach with my preservice teachers that involves an alternate base (five) and new symbols as well as new language for the alternate base (see Hopkins & Cady, 2006). The approach uses the idea presented here; we create a term for a group of five (*flop*) and for five groups of five (*flip*). I have found it remarkable that my students have to think very hard to answer the question *how many flops in a flip?* even after substantial experience counting out loud, grouping objects, drawing pictures and discussing relationships between this new number system and the decimal system. Preservice elementary teachers who struggle to understand why a child might not know that there are ten tens in one hundred can understand the complex task of learning the decimal system when they notice how challenging questions about the relationship between *flop* and *flip* is for them. And they understand that missteps in writing symbols for number words, as they see Talecia making in the video, are an essential part of making sense of a number system.

Summary

We have seen five ways to read 200_{four} aloud. It is important to remember that each of these possibilities is based on a principled argument about place value numeration. These arguments may be explicit or implicit, but each possibility results from the application of human intelligence to the problem of communicating quantities.

Mathematics content courses for intending elementary teachers (in the English speaking world) represent a particular context for solving this problem. In this context, it seems important to address issues of language and numeration in our place value instruction. If we do not, then we present learning the decimal system as unproblematic and straightforward, which it is not. If we do not, then we do little to change our future teachers' own conceptions of place value, which we know to be problematic. If we do decide to make understanding the language of number an objective of our place value instruction with future teachers then an approach that names the relevant grouping and that names groups of these groups is promising.

Notes

[1] These possibilities are the result of a year's worth of extended thought and conversation about place value with colleagues, students and native speakers of other languages. As a group, they exhaust my own understanding of place value. I hope to have my thinking stretched by readers who see new ways to verbalize 200_{four}.

[2] See R. A. Philipp's unpublished manuscript, *Motivating prospective elementary teachers to learn mathematics by focusing on children's mathematical thinking.*

[3] My not-quite-four-year-old son Griffin asked me, *Is there a teen with a two in it?* After praising the cleverness of the question, I replied, *No. But if there were, it would be twelve. If there were a teen with a one in it, what number would that be?* He suggested talking about something else.

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