THE REFLECTIVE CYCLE OF STUDENT ERROR ANALYSIS

JOHN LANNIN, BRIAN TOWNSEND, DAVID BARKER

> Give me a productive error over a boring, mundane and unproductive fact any day. (Anon. [1])

Current instructional theory encourages the use of errors as "springboards" for further learning (Borasi, 1987). Despite the instructional focus on dealing with and avoiding student errors, little attention has been given to the process students use to reconcile their errors. We examined the ways in which students recognized, attributed, and attempted to reconcile their errors through the use of a conceptual framework, the *reflective cycle of error analysis*. We studied the reasoning of two twelve-year old students as they attempted to generalize numeric situations, describing the factors that contributed to student error recognition, the causes to which students attributed their errors, and the strategies they used to reconcile their errors. We emphasize the importance of making the process of reconciling, attributing, and reconciling errors transparent to students in the classroom.

A marked shift has occurred in the way student errors are viewed by mathematics teachers and students. Initially, teachers were to follow a behaviorist view of learning, providing positive reinforcement when students yielded correct answers and initiating negative reinforcement or withholding positive reinforcement when attempting to eliminate errors (Miller, 1983). As part of this process, teachers were encouraged to "diagnose and remediate" errors through a repetitive stimulus-response process. However, recent instructional recommendations encourage teachers to view student errors as attempts to make sense of mathematical ideas. In accordance with this view, errors are to be used as "sites for learning" (Hiebert, Carpenter, Fennema, Fuson, Murray, Olivier, and Human, 1997), wherein the teacher uses errors as opportunities for students to delve deeper into conceptual issues (Borasi, 1996). A contrasting instructional perspective has been developed regarding student errors, one that places greater emphasis on students reflecting on and learning from their errors. However, we know little about how students recognize and resolve the errors they make. We discuss a framework for examining the process that students utilize for recognizing and reconciling their errors.

Background on student errors

Early education researchers (*e.g.*, Buswell and Judd, 1925) were aware of a variety of student mathematical errors. Continued research in this area has led to the recognition of a variety of resilient errors in various mathematical domains (*e.g.*, Clement, 1982; Cooper, 1992; Englehardt, 1982; Roberts, 1968). However, differences have emerged in how our knowledge of these errors can be used in the classroom.

Early instructional strategies involved extinguishing errors through direct correction and remediation. An illustration of a teacher's attempt to eradicate an error is found in Buswell (1926) where the teacher describes how she dealt with her students' difficulties with subtraction,

> In the case of the process of subtraction I found that some pupils in the class read [subtraction expressions] the problems backward, so I explained the meaning of the term 'subtract' and showed them that the number subtracted was always smaller. Then I gave them groups of examples in which they merely read the example correctly." (p. 190)

In this example, the role of the teacher attempted to eliminate student confusion through repetition of the 'correct' view of subtraction.

Recently, researchers (Novak, 1987; Novak and Helm, 1983), aided by a new perspective of how students learn, have viewed errors differently – noting that student examination of errors could aid understanding. However, engaging students in the process of examining their errors is not without challenges. For example, when Papert (1980) observed students attempting to construct programs using the LOGO programming language, he noted that students often operate in unproductive ways:

> A program is quickly written and tried. It doesn't work. Instead of being debugged, it is erased. Sometimes the whole project is abandoned. Sometimes the child tries again and again and again with admirable persistence but always starting from scratch in an apparent attempt to do the thing "correctly" in one shot. […] The ethic of school has rubbed off too well. What we see as a good program with a small bug, the child sees as "wrong," "bad," "a mistake." School teaches that errors are bad; the last thing one wants to do is to pore over them, dwell on them, or think about them. (pp. 113-114)

Such negative views of errors inhibited student success when attempting to write LOGO programs, as they appeared unwilling to reflect on the potential sources for their errors, often choosing to erase and start over.

Further understanding of how students consider their errors is necessary to guide instruction related to errors. We discuss how two students in our study examined the errors they committed during an eighteen-week teaching experiment focused on developing algebraic reasoning. In our study, pairs of twelve-year-old students were provided with tasks to provoke the development of generalizations. During this process, the students periodically made errors and attempted to resolve them. As we analyzed these instructional sessions, we realized that we needed to understand better the reasoning process that students engaged in as they made errors. We, therefore, identified differing ways that students dealt with their errors. We focus our attention on Dallas and Lloyd, two students who often struggled with the use of proportional reasoning.

The reflective cycle of error analysis

In order to investigate student cognition related to errors we developed and revised a conceptual framework that we call the *reflective cycle of error analysis* (see Figure 1) to inform our investigation of student errors. This framework describes the process that students move through as they examine errors that they make.

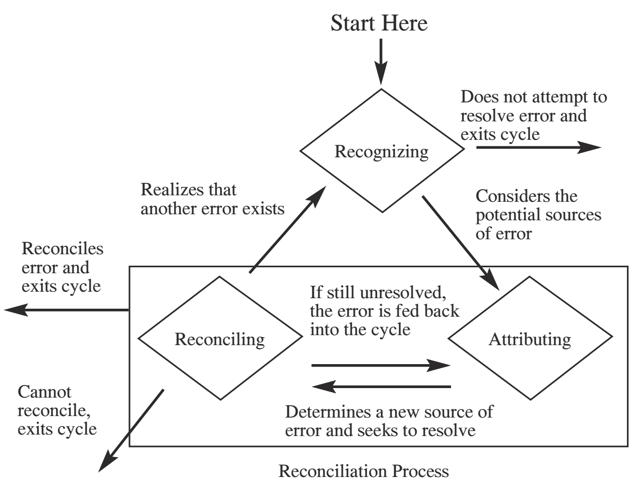


Figure 1: The reflective cycle of error analysis.

Starting at the top box, a student recognizes that an error has occurred and chooses to either ignore the source of the error or consider potential sources for the error. Once the error has been attributed to a particular source or sources, the student attempts to reconcile the error to eliminate the cognitive conflict that he or she experiences. After implementing a strategy to reconcile the error, the student can deal with the error in one of three ways:

- successfully reconcile the error and exit the cycle
- stop the reconciliation process without resolving the error and exit, or
- recognize that the original source of attribution of the error is incorrect and seek to other potential sources of error.

We examine three important questions related to how students dealt with the cognitive conflict they experienced as they examined their errors:

What factors contributed to student recognition of errors?

To what potential causes did students attribute errors?

What strategies did students use to reconcile errors?

In the following sections, we illustrate the components of recognizing, attributing, and reconciling errors. We devote much of our attention to the reconciliation process, as we view this as a critical part of students' performance when dealing with errors.

Factors that contributed to student recognition of errors

In our examination of student errors we considered all situations where students appeared to recognize an error and entered the reflective cycle of error analysis. We identified three categories under which all instances involving students recognition of errors were placed.

Discrepant results: when a single student employed two different strategies, generating differing results, for the same instance.

Social influence/explaining a rule to others: via a statement regarding the error by another person; a question or prompt from an outside person; or listening to another student's explanation.

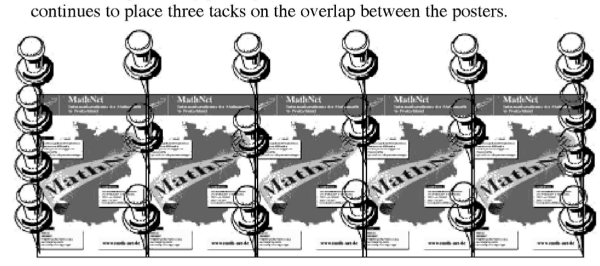
Unreasonable results: when their strategy generated a value that was, from their perspective, too large or too small.

Potential causes to which students attributed errors

Once students identified an error, they generally considered a potential source or sources for their errors. Considerable variation existed in the sources identified and examined for their errors. The students in our study attributed their errors to at least one of five causes.

Calculations: Students frequently identified calculations as a source of error. An example of this occurred during the *poster problem* (see Figure 2) when Lloyd attempted to calculate the number of tacks for 20 posters by doubling the number of tacks required for 10 posters (35) to arrive at 70 tacks. After drawing a diagram and counting the 65 tacks required for 20 posters Lloyd decided to verify his solution for his doubling strategy by checking his calculations with the calculator, entering 35 + 35 and arriving at the same incorrect result. As he continued working on this error, Lloyd questioned the calculations performed by the technology, noting that the calculators "times before adding" and that "sometimes calculators get confused." At this point, Lloyd appeared to believe that his doubling strategy would

Deck R. Ater has decided that three tacks is not enough. He decides to reinforce his posters by placing four tacks on each of the ends. He continues to place three tacks on the overlap between the posters.



1. How many tacks will Deck need to hang 3 posters? 6 posters? 10 posters? 20 posters? 27 posters? 35 posters?
2. Explain how you would determine the number of tacks for any number of posters.

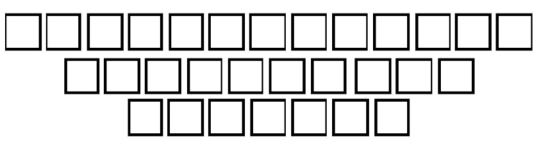
Figure 2: The poster problem.

generate the correct result and attributed his error to mistaken calculations.

Rules: The open-ended nature of the tasks allowed students to develop their own means for generating solutions, leading to the development of rules that they questioned. This often occurred when Lloyd or Dallas used a guess-and-check method to develop a rule. For example, in the *theater seats problem* (see Figure 3), Lloyd found that the 5th row had 28 seats and guessed that the rule was $n \times n + 3$. After trying the rule for the 6th row, Lloyd concluded that his rule was incorrect. Dallas commonly used this "guess-and-check" strategy, and, therefore, attributed many errors to his rules as well. For example, while working the *cube sticker problem* (see Figure 4), Dallas produced the rule $n \times 5$ after noticing that a rod of length 2 required 10 stickers. He then attempted to verify his rule for a length-3 rod and noticed that the rule produced an incorrect result for this instance. When attributing an error to a rule, Dallas and Lloyd focused on product (the rule) rather than the process (strategy) used as the source of error.

Strategy: Over the course of the 18 weeks, Dallas and Lloyd moved from focusing on their rules to examining the strategies they used to construct their rules. An example of this occurred when Dallas attempted to apply his "guess-and-check" strategy for various problem situations. As noted in the paragraph above, Dallas applied this strategy when attempting to find a rule for the *cube sticker problem*. In a later session, Dallas stated that such a strategy was not as effective as thinking about the context because before he was using, "pure math [guessing-and-checking], and now I am more doing what I seeing in front of me [by connecting to the diagram]". In this instance, Dallas recognized that his way of operating was not productive and created errors in the rules he constructed.

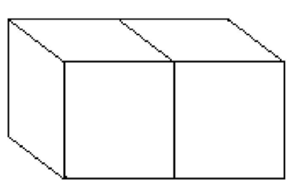
In a theater there are 7 seats in the first row. The increase in the number of seats is the same from row to row. Below is a diagram of the first three rows in the theater.



1. How many seats are there in the 4th row of the theater? In the 5th row? In the 10th row? In the 23rd row? In the 24th row? In the 38th row?
2. Explain how you would determine the number of seats in any row. Write a rule that would allow you to calculate the number of seats in any row. Explain your rule.

Figure 3: The theater seats problem.

A company makes colored rods by joining cubes in a row, using a sticker machine to place stickers on the rods. The machine places exactly one sticker on each exposed face of each cube. Every exposed face of each cube must have a sticker, so this length two rod would need 10 stickers.



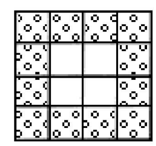
1. How many stickers would you need for rods of length 7? Length 10? Length 49? Explain your reasoning,
2. Explain how you could find the number of stickers needed for a rod of any length.

Figure 4: The cube sticker problem.

Syntax: Dallas and Lloyd sometimes attributed their errors to the syntax they used when recording their generalizations in symbolic form or when using technology. These errors occurred as a result of their need to communicate their thinking with another person or when using the calculator or computer spreadsheet.

For example, Dallas verbalized his rule for the *border problem* (see Figure 5) by stating that he computed the total area of the square and subtracted away the area of the inner square [We could represent his rule as: $n \times n - (n - 2)(n - 2)$ where n is the length of the border]. However, when attempting to write his rule, he began by writing $n \times e$ for the area of the entire figure. When asked what $n \times e$ meant, he stated that both n and e described the length of the border and changed his expression to $n \times n$. We note that this is not an error in understanding from Dallas' perspective. Dallas recognized that to communicate clearly what he intended for this situation, he must write the expression so that the same quantity was symbolized in the expression $n \times n$. Despite his correct verbal description of his rule, Dallas exhibited difficulty translating his rule into formal algebraic symbols.

The Washington Student Council is creating designs with a dotted pattern on the border. The council would like to know how many squares are needed with the dotted pattern. They have asked the 5th grade class for help.



1. How many squares are in the border of a 4 by 4 grid? A 7 by 7 grid? A 10 by 10 grid? A 16 by 16 grid? A 25 by 25 grid? A 100 by 100 grid?
2. Write a rule to find the number of squares in the border of any size grid.

Figure 5: The border problem.

Non-specific errors: As Dallas and Lloyd recognized errors, they periodically appeared unable or unwilling to attribute their errors to a specific cause. In these situations, they ignored the source of their errors or 'invented' a reason for their errors. Such unwillingness to grapple with the source of these errors often caused them to repeat the same error later on.

An example occurred when Dallas attempted to find the number of seats in the 50th row for the *theater seats problem* (see Figure 3). Dallas incorrectly computed the number of seats in the 23rd row, resulting in a total of 77 seats. When asked to find the number of seats in the 50th row, Dallas doubled the number of seats in the 23rd row (incorrectly assuming that this generated the correct number of seats in the 46th row) and added 4 more groups of three for the remaining four rows, obtaining a total of 166 seats. When Lloyd shared his incorrect strategy of multiplying the number of seats in the 10th row (34) by 5 to obtain a result of 170, Dallas appeared to invent a reason for the four-seat discrepancy between his result and Lloyd's result. Dallas said that he failed to add an "extra four," though he was unable to elaborate on where these four seats were located.

Strategies for reconciling errors

After recognizing an error and attributing their errors to at least one particular source, Dallas and Lloyd attempted to reconcile their errors. As such, the goal was usually to understand the source of the error so that any cognitive conflict was removed. However, some reconciliation strategies did not appear to assist them in deepening their understanding of the source of their errors, often leading to repeating the same error. We identified three strategies that Dallas and Lloyd used to resolve their errors:

Recalculating or adjusting previous calculations: When Lloyd and Dallas attributed their errors to their calculations, a checking of the calculation or a recalculation was often performed. In some cases, the recalculation served as verification for the original method of calculation (such as the student's computational skills) by doing the same calculation through other means (such as using a calculator or spreadsheet technology). For example, when working with the *poster problem* (see Figure 2), Lloyd incorrectly doubled the number of tacks for 10 posters to find the number of tacks for 20 posters. After recognizing that this generated an incorrect number of tacks, Lloyd performed the same calculation again by hand, with a calculator, and on his computer spreadsheet.

At times, Dallas and Lloyd reconciled their errors by simply adjusting calculations to match the result they expected. When they employed this method, it appeared to be an attempt to avoid thinking deeply about the error that occurred. Instead, this strategy involved a simple adjustment of their results. Such action in reconciling an error often led to repeating the error or misapplying a strategy to different situations.

An example of the *adjusting calculations strategy* occurred when Lloyd attempted to calculate the number of tacks needed for 20 posters for the *poster problem* (see Figure 2). Lloyd implemented two strategies for determining the number of tacks. The first (and incorrect) strategy was to double the number of tacks needed for 10 posters, 35, resulting in 70 tacks. Lloyd's second strategy involved repeatedly adding 3 tacks for each additional poster attached to the initial poster (that required eight tacks). Lloyd used his computer spreadsheet to arrive at the correct result of 65 tacks for 20 posters. However, soon after this amount was displayed on the computer spreadsheet, Lloyd manually replaced 65 with 70 in the cell for 20 posters (see Figure 6). When questioned about the jump from 62 tacks for 19 posters to 70 tacks for 20 posters, Lloyd stated that eight tacks were needed for the ends of the posters. Lloyd reconciled the discrepancy between his results by adjusting the value to the one that he thought was correct.

Modeling/reinterpreting the situation: Lloyd and Dallas often reconciled their errors by modeling the situation and/or reinterpreting the strategy in relation to the problem context.

	A	B	C
1	Lloyd		
2			
3	Number of Posters	Number of Tacks	
4	1	8	
5	2	11	
6	3	14	
7	4	17	
8	5	20	
9	6	23	
10	7	26	
11	8	29	
12	9	32	
13	10	35	
14	11	38	
15	12	41	
16	13	44	
17	14	47	
18	15	50	
19	16	53	
20	17	56	
21	18	59	
22	19	62	
23	20	70	
24	21	73	
25	22	76	
26	23	79	
27			
28			
29			
30			

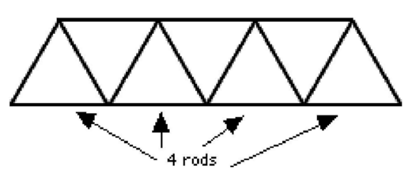
Figure 6: Lloyd's spreadsheet for the poster problem.

This generally occurred in one of two ways:

- the student attributed the error to a misrepresentation and hence reconciled it by reinterpreting the situation, or
- the student attributed the error to a strategy and used the problem context or modeling as a tool to uncover the mistake.

For example, when working the *beam problem* (see Figure 7), Lloyd used an incorrect strategy. He doubled the result for a length-5 beam (39), to obtain a total of 78 rods for a

Beams are designed as a support for various bridges. The beams are constructed using rods. The length of the beam is determined by the number of rods used to construct the bottom of the beam. Below is a beam of length 4.



1. How many rods are needed to make a beam of length 5? Of length 8? Of length 10? Of length 20? Of length 34? Of length 76? Of length 223?
2. Write a rule or a formula for how you could find the number of rods needed to make a beam of any length.

Figure 7: The beam problem.

length-10 beam. With the help of Dallas, Lloyd drew a picture of the situation and realized that an additional rod was necessary to connect the two length-5 sections. By modeling the situation, Lloyd appeared to gain further insight into why his strategy did not produce the correct result. He was able to modify his initial strategy, adding an additional rod to obtain the correct answer.

Abandoning the rule: Another strategy that Lloyd and Dallas used to reconcile their errors involved abandoning their rules without critically examining why the strategy may or may not have been in error. Typically this happened in one of two settings:

- when the student employed a guess-and-check strategy when attempting to find a general rule
- in situations where the student was more confident using a different strategy.

This strategy for reconciling errors did not provide insight into the source of the error that occurred.

Abandoning a rule occurred most often with Dallas. For example, when attempting to find a general rule for the number of stickers Dallas noticed that a length-2 rod required 10 stickers for the *cube sticker problem* (see Figure 4). He conjectured that the rule could be "multiply by 5" and tested this rule for a length-3 rod. After creating a model for a length-3 rod and counting the number of stickers, he found that the "multiply by 5" rule did not provide the correct number of stickers and discontinued the use of his "multiply by 5" rule. When the guess-and-check strategy was implemented, both Dallas and Lloyd were willing to abandon quickly rules that did not generate correct results.

An example of abandoning a rule when students were more confident in another strategy occurred when Dallas attempted to double the number of seats in the 5th row of the theater, 19, to find the number of seats for the 10th row for the *theater seats problem* (see Figure 3). After arriving at the incorrect result of 38 seats, Dallas shared his result with Lloyd. Lloyd stated that he repeatedly added 3 on his calculator from the number of seats in the 5th row, generating the result of 34 seats. At this point, Dallas said that his strategy was incorrect and tried to repeatedly add 3 to find the value for the next instance.

Discussion

The ways that Dallas and Lloyd dealt with their errors have important instructional implications. The factors to which they attributed their errors and the strategies they used to reconcile their errors led to differences in how they viewed their errors and their subsequent repetition of these errors.

One issue that arose early in the study involved situations in which Lloyd and Dallas attributed their mistakes to what we characterized as *non-specific errors*. Instead of attempting to determine particular sources of errors, their notion that an error had occurred "somewhere" caused them to discontinue their efforts and exit the reflective cycle before reconciling their errors. However, later in the study, when they attributed their errors to specific causes, such as their rules or strategies, they began a process that had the potential to uncover their errors and deepen their understanding of why errors occurred. As suggested by Borasi (1987), teachers could discuss the process of resolving errors, highlighting the potential for exploring errors to lead to deeper understanding.

Another challenge for Dallas and Lloyd was considering the many potential sources for error that existed when working in a problem-solving environment. For example, since Lloyd examined only his calculations as the source of error for the *poster problem* he ignored another possible source for error, his use of the doubling strategy. Dallas and Lloyd often had difficulty considering the process that they used to generate values or rules as a potential source for error. They tended to focus on the calculations or rules they created rather than the strategies they applied to generate particular values. Teachers can facilitate student awareness of the various sources for errors by encouraging students to discuss potential sources for their errors, emphasizing an examination of the strategies that students use as a potential source for error.

Similarly to what Battista (1999) found when students were attempting to enumerate three-dimensional cube arrays, we found that students often "invented" strategies in an attempt to reconcile their errors. In his study, Battista noted that students adjusted their results to arrive at the desired correct value without delving deeper into why such an adjustment could be made. Lloyd demonstrated similar reasoning when grappling with the discrepant results he generated when reasoning recursively and using the doubling strategy for the *poster problem*. Lloyd appeared to have difficulty recognizing the generality of his application of proportional reasoning for the *poster problem*. He was able to see that he could not double certain instances, but did not recognize that the same reasoning applied to other values, such as jumping from 50 posters to 100 posters. Such general reasoning has often not been the focus of mathematics instruction despite the importance of generalizing within mathematics (Mason, 1996). Investigating similar tasks where Dallas and Lloyd continued to examine the use of proportional reasoning certainly aided their understanding of their errors. Encouraging them to reconsider the use of proportional reasoning across tasks and over an extended period of time appeared to impact their progress in dealing with this error. However, we still need to understand better how students view the general nature of their errors.

Although our study focused on the reflective cognition of two students, we note the important influence of the social environment in the teaching experiment. We consistently asked Dallas and Lloyd to explain why they could or could not use particular strategies. As such, this became a normative part of our work for these tasks. We recognize that each student individually shapes and adapts the way he or she views errors; however, this change occurs, in part, due to the accepted norms that are developed for operating within the classroom (Davis and Simmt, 2003). These norms are dynamic, impacting the way individuals and groups of students react to errors. A classroom environment in which students share and reflect on their reasoning with others is essential for successful reconciliation of errors. Davis and Simmt emphasize that interacting with ideas is the key component in the process of learning, stressing that the critique of ideas can further emphasize the role of reasoning to develop a sense of the correctness of answers. During our study, Dallas and Lloyd moved away from unproductive ways of dealing with errors, such as attributing errors to non-specific causes, toward identifying strategies and rules that led to their errors. As they discussed and were questioned about their errors, the social culture influenced their ways of operating with these errors.

We analyzed how two students recognized their errors, the factors they attributed their errors to, and the strategies they used to reconcile these errors using the framework we call the *reflective cycle of error analysis*. As such, this framework builds upon the instructional recommendations of Hiebert *et al.* (1997) and Borasi (1987) by providing insight into the ways students address the errors they recognize. Current instructional recommendations place increased emphasis on assisting students in becoming autonomous problem solvers who can diagnose and use their errors in productive ways. Our framework can provide guidance to teachers in using student errors to deepen understanding of mathematical ideas. The process of recognizing, attributing, and reconciling errors is a complex one that has often been underemphasized in the mathematics classroom. However, this process is critical in many careers where mathematics plays an important role, such as the domains of computer science and engineering. We must further understand how we can encourage students to make use of their errors productively rather than simply to ignore them.

Notes

[1] Quotation attributed to "Anon." found at: http://www.worldofquotes.com/topic/Error/1/index.html, accessed 25th August, 2006.

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[These references follow on from page 29 of the article "Teaching for understanding for teaching" that starts on page 24 (ed.)]

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