ON RELATIVE AND ABSOLUTE CONVICTION IN MATHEMATICS

KEITH WEBER, JUAN PABLO MEJIA-RAMOS

> What is a proof? Well, it is an argument that convinces someone who knows the subject. (Davis & Hersh, 1981, p. 40)

> Proof is not necessarily a prerequisite for conviction—to the contrary, conviction is far more often a prerequisite for finding a proof. (de Villiers, 1990, p. 18)

What types of mathematical arguments do students find convincing? What types of mathematical arguments *should* students find convincing? Much contemporary research on justification and proof can be viewed as addressing these two questions (see Harel & Sowder, 2007). In this article, we argue that the meaning of these two questions is ambiguous, due to two possible interpretations of the word "convince", which we term *absolute conviction* and *relative conviction.* This ambiguity has led to unwarranted claims and disagreements in the mathematics education literature. We interpret absolute conviction to mean that a student who is convinced of a claim has absolute certainty that the claim is true. Relative conviction refers to the idea that when a student expresses conviction in a claim, the student is expressing that the probability that they would assign to a claim exceeds a certain threshold. In this article, we question whether the notion of absolute conviction provides a useful lens for analyzing students' proof-related behavior. Instead, we contend that relative conviction is more common. The idea that accepting an argument or a claim is a matter of evaluation and judgment involving probabilities can provide a more accurate and nuanced account of students' behavior on proof-related tasks and can result in more appropriate goals for mathematical instruction. In particular, if students prefer empirical arguments to deductive ones or are not fully convinced by a deductive argument, they do not necessarily have a deficient view of proof.

Theoretical assumptions

There is an assumption amongst many mathematics educators that a primary goal of mathematics instruction is for students to adopt the standards for proving and conviction that mathematicians hold. Due to this assumption, many studies in mathematics education fit the following description:

1. students are given a justification-related task;
2. by analyzing students' behavior while completing this task, researchers infer students' beliefs about justification and proof;
3. the researcher compares these beliefs with those of the mathematical community;
4. discrepancies between students' and mathematicians' beliefs are identified;
5. these discrepancies are then used as a call for researchers to design instruction that leads students to transform their beliefs about justification and proof to those held by mathematicians (see Harel & Sowder, 2007).

Since researchers are measuring the distance between students' and mathematicians' views on justification and proof, their evaluations are dependent on assumptions of what mathematicians' views are with respect to justification and proof. We discuss three such assumptions below. To avoid ambiguity, we first explicitly define some basic terminology.

A *mathematical statement* is a precise assertion situated within a broader mathematical theory that is assumed to have a truth-value. For example, "the square of an odd integer is an odd integer" is a mathematical statement (within the context of a theory consistent with our interpretation of arithmetic) that has been shown to be true. The scope of a mathematical statement is the set of objects to which the statement pertains. An *empirical argument* supports a mathematical statement by showing that the statement holds true for a proper subset of the scope of that statement. For instance, the claim that "the square of an odd integer is also an odd integer" has a scope of the set of all odd integers. If a student argued that this claim was true by verifying it with a finite collection of odd integers, this would be an empirical argument. A *deductive argument* in support of a mathematical statement is a sequence of assertions that concludes with the mathematical statement where each assertion in this argument is either a claim known or assumed to be true or purports to be a logically necessary consequence of previous assertions [1]. A deductive argument might contain a logical error. If so, we call this argument *invalid*. If the deductive argument contains no errors, we call it *valid*.

Below are three assumptions about mathematical practice common in the mathematics education literature. We are neither claiming that all mathematics educators subscribe to these assumptions nor do we endorse these assumptions ourselves. Instead, as we illustrate, we only claim that these assumptions are commonplace in the literature and that such assumptions are often stated without supporting argumentation.

- *Assumption 1.* To mathematicians, a proof is a valid deductive argument.

- *Assumption 2.* In mathematical practice, empirical arguments are recognized as inherently limited and unreliable. Consequently, an empirical argument in support of a mathematical statement is insufficient to convince a mathematician that the statement is true.
- *Assumption 3.* Given a mathematical statement S, if a mathematician is convinced that there is a valid deductive argument (*i.e.*, a proof) of S, then that mathematician will be convinced that S is true. Seeking further confirmatory evidence for S would be superfluous.

To avoid misinterpretation, we do not adhere to these assumptions. Indeed, we give extended arguments against Assumption 2 and Assumption 3 elsewhere (Weber, Inglis, & Mejía-Ramos, 2014) and in this article, we argue against Assumption 3. We are merely claiming that these assumptions undergird much of the research with regard to justification and proof, and so we will use them as a starting point for our discussion.

Inconsistencies in the literature on conviction and proof

There is a theoretical inconsistency in the mathematics literature on proof [2]. As the quotations at the beginning of the article illustrate, some researchers view proving as tantamount to convincing (*e.g.*, Balacheff, 1987), while others argue that there are convincing arguments that are not proofs (*e.g.*, Tall, 1989) and that a proof might not be fully convincing (Morris, 2002).

Researchers have typically tried to demonstrate that students' behavior is inconsistent with Assumption 2 about mathematical practice (that mathematicians do not obtain conviction via empirical arguments), either by showing that students will justify a mathematical statement with an empirical argument or by showing that students will claim to be convinced by an empirical argument that they have been shown. For instance, Recio and Godino (2001) asked about 400 first-year university students to prove one statement from algebra and one from geometry; they found that roughly 40% of students' responses consisted of empirical arguments. However, some have questioned these types of interpretations. Weber (2010) argued that it is risky to make inferences about students' beliefs from the incorrect work they hand in; students may submit fallacious arguments not because they thought they were right, but to please the interviewer or to obtain partial credit. Stylianides and Stylianides (2009) found that most pre-service teachers who submitted empirical arguments as justification of mathematical statements actually did not think their arguments were correct. This is not to say that no students are convinced by empirical arguments, only that we question whether students' behavior on justification tasks is a sufficient warrant to establish this claim.

Other researchers have attempted to demonstrate that students' behavior is inconsistent with Assumption 3 about mathematical practice (that valid deductive arguments in support of a statement provide mathematicians with complete conviction that the statement is true). Morris (2002), for instance, found that many pre-service elementary school teachers claimed that they remained uncertain that a mathematical claim was true after reading a valid deductive argument that supported the claim. However, Stylianides and Al-Murani (2010) conducted interview studies with 16 secondary school students who they judged as most likely to believe that a proof and a counterexample to the same theorem could exist (in an earlier survey, they judged both a proof and a counterexample to the same statement as likely to receive high marks from their teacher). They found that none of these students believed a theorem that had been proven could later be refuted with a counterexample. Again, this is not to claim that there are not students who do not find deductive arguments to be convincing, only that Stylianides and Al-Murani's study suggests we may be overestimating the extent to which this belief is held.

What does it mean to be convinced? Absolute and relative conviction

We believe that the inconsistencies in the literature described above concern an inherent ambiguity in what it means to be convinced. In particular, we believe there are two types of conviction.

> *Absolute conviction:* Reed (2008) argued that although absolute conviction, or certainty, is recognized by many to be an important construct in philosophy and psychology, there does not exist an analysis of certainty that is not contentious. For us, an individual who has absolute conviction in a claim has a stable psychological feeling of indubitability about that claim; we do not claim that the individual's conviction is necessarily epistemologically warranted.
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> *Relative conviction:* An individual has relative conviction in a claim if the subjective level of probability that one attributes to that claim being true exceeds a certain threshold to provide a warrant for some future actions.

We contend that mathematicians have absolute conviction in some claims. For instance, we believe that mathematicians have absolute conviction in the validity of *modus ponens.* We suspect most mathematicians have absolute conviction in other claims such as 2 + 2 = 4, every real number has an additive inverse, and so on. Whether a mathematician has absolute conviction for more sophisticated claims depends on her epistemic stance and experience. Our key point is that mathematicians also have relative conviction in the truth of some claims and the validity of some proofs, *including claims that have been proven.* For instance, in 2005, the *Annals of Mathematics* published Hales's proof of the Kepler Conjecture even though the journal's editor said the reviewers were only 99% sure the proof was valid. That is, the editor and team of reviewers conceded they did not have (and felt they could not obtain) absolute conviction that the proof was valid, but their conviction in the validity of the proof was sufficiently high as to provide a warrant for publishing the proof in this prestigious outlet [3].

In everyday discourse, the word "convinced" usually refers to relative conviction. For instance, consider a news report from the Reuters website that begins with the headline "Scientists are more convinced mankind is cause of [global] warming" [4]. In the article, the authors cite a survey showing that most climate scientists can state that mankind is contributing to global warming with 95% confidence. The use of 95% confidence explicitly indicates relative conviction. More importantly, the phrase "more convinced" would not make sense with absolute conviction. If one is absolutely certain, one cannot be *more* absolutely certain. However, with relative conviction, one can be "more convinced" if one's confidence in a statement exceeds a higher threshold.

The connection between absolute conviction and proof

Some mathematics educators have reasoned that what makes proof special in mathematics is its potential to provide mathematicians with absolute conviction. Fischbein and Kedem (1982) asserted that "a formal proof confers on it *a priori* universal validity" (p. 128), a stance consistent with Assumption 3 that mathematicians consider performing subsequent checks on a proven statement to be superfluous. However, Fischbein (1982) also found that secondary school mathematics students still desired to verify a statement with examples after reading its proof, which seems to be inconsistent with this assumption. Fischbein attributed the students' desire for more evidence to them having only formal conviction for a proof, not the type of intuitive conviction that leads to absolute conviction. He concluded that what is needed is *"a complimentary intuitive acceptance of the absolute predictive capacity of a statement which has been formally proved"* (p. 18, italics in original). Fischbein argued that students need extensive experience with proof in order to understand the validity of proof intuitively.

Harel and Sowder (1998) introduced the notion of proof schemes, which involved understanding how students turned conjectures into facts: that is, how they sought and obtained absolute conviction in mathematical assertions. Unlike Fischbein, Harel and Sowder emphasized that for mathematicians, mathematical truth is, in principle, couched within the axiomatic system in which they are working, but believed that proof could provide absolute conviction relative to that system. Harel and Sowder focused the field's attention on students' methods of obtaining absolute conviction. This work has led to the designing of instruction with the following aims: students should reconsider their stance that empirical evidence and appeals to authority are reliable ways to obtain mathematical truth and students should treat deductive evidence as a legitimate means of obtaining absolute conviction.

Dreyfus (2004) raised two challenges for the proof schemes research program. First, if we want students to adopt mathematicians' standards of conviction, then we need to identify a universal frame of reference held by all mathematicians. Harel and Sowder (2007) were explicit on this point, noting that their proof schemes framework is dependent on the fact that a shared proof scheme amongst mathematicians exists. Dreyfus questioned this assumption by claiming that there are many types of proofs, such as computer-assisted proofs and visual arguments, whose validity is contentious amongst mathematicians. Second, Dreyfus noted that mathematicians perhaps should not accept that any form of proof can confer absolute certainty, citing issues such as the fact that the consistency of the axiom systems in which we work has not been established (and is indeed unproveable) and Lakatos's (1976) arguments for mathematical fallibility [5]. In the remainder of this article, we go further in challenging this research by arguing that (i) *even if* mathematicians accept that deductive arguments can establish that a mathematical claim is true with certainty, they still might lack confidence in proven statements and (ii) *even if* mathematicians do not obtain absolute conviction from empirical arguments, it does not follow that students are behaving inconsistently with mathematicians if they claim to be convinced by such arguments.

Re-interpreting our assumptions about mathematical practice

When listing the three assumptions about mathematical practice, we did not distinguish between absolute conviction and relative conviction. We argue that for these assumptions to accurately capture mathematical practice, the references to conviction must be referring to absolute conviction. To illustrate why this must be the case with Assumption 2, consider the following quotation by Pólya (1957):

> Having verified the theorem in several particular cases, we gather strong inductive evidence for it. The inductive phase overcame our initial suspicious and *gave us strong confidence in the theorem*. Without such confidence we should scarcely find the courage to undertake the proof. (pp. 83-84, emphasis added)

Pólya's statement suggests that inductive evidence can provide us with *relative conviction* in a theorem, conviction that he deemed *necessary* to do mathematics. Consequently, we believe that one should not be bothered if students obtain relative conviction via an empirical argument. Students are only behaving inconsistently with Assumption 2 if they gain absolute conviction from an empirical argument.

Inglis, Mejía-Ramos and Simpson (2007) made a similar point. An argument to support a claim may have a qualifier specifying the level of confidence that the claim is true (*e.g.*, possibly, probably, definitely). A distinction between empirical arguments and deductive proofs is the qualifier that should be attached to them. Empirical arguments should have a non-absolute qualifier (*e.g.*, based on this empirical argument, this statement is *probably* true) while proofs should not be qualified in this way (*e.g.*, based on this proof, this statement is *definitely* true). Faults in students' reasoning can be conceptualized as students applying an inappropriate qualifier to an empirical argument or proof. A key point is that researchers should not *infer* what qualifier students are using if they do not say it; rather researchers should actively seek out these qualifiers (see Inglis & Mejía-Ramos, 2008).

The preceding analysis overlooks an important possibility. Paseau (2011) argued:

> That we are in possession of a proof of p does not imply we should be certain of p [...] The proof may be long and hard to follow, so that any flesh-and-blood mathematician should assign a non-zero probability to its

being invalid. The longer and more complex the proof, the less secure its conclusions. (p. 143)

Note that Paseau is not claiming that *proof* can *never* provide absolute conviction in a theorem (in the psychological sense described earlier), but he is saying that proofs that are long and hard to follow ought not to provide this conviction for human mathematicians. Given that students frequently have difficulty in determining if a proof is valid, it makes sense that they may only obtain relative conviction in the validity of the proof and, consequently, in the truth of the theorem. Thus it is reasonable for students to seek confirming evidence in a mathematical statement after reading a proof of that statement. To behave inconsistently with Assumption 3, students would need to have absolute conviction that a proof of a theorem was correct, but lack absolute conviction that the theorem was true.

De Villiers's (1990) claim that "proof is not necessarily a prerequisite for conviction—to the contrary, conviction is far more often a prerequisite for proof" (p. 18) is not inconsistent with Davis and Hersh's (1981) contention that to mathematicians, a proof "is an argument that convinces someone who knows the subject" (p. 40). If we interpret de Villiers as using a relative conception of conviction and Davis and Hersh as using an absolute one (or perhaps a relative one with a higher threshold for conviction), the two statements are complementary. Indeed, both would be consistent with the Pólya quotation above. An obvious consequence is that authors should always distinguish between absolute and relative conceptions of conviction when they use "convince" and related terms.

We suspect that researchers overestimate the extent to which students' and mathematicians' conceptions of justification and proof differ. If students say they are convinced an assertion is true but do not explicitly qualify their conviction, researchers sometimes assume that students are speaking of absolute conviction. However, as we noted, in everyday discourse, conviction usually denotes relative conviction. Hence, the assumption that a student's unqualified use of conviction implies that the student refers to absolute conviction is unwarranted (see Inglis & Mejía-Ramos, 2008).

Illustrations of qualitative analysis using relative conviction

In the previous section, we argue that the distinction between absolute conviction and relative conviction is necessary to interpret claims about mathematical practice and students' mathematical behavior. In this section, we illustrate how keeping this distinction in mind can provide a different interpretation of students' utterances than has appeared in the research literature. We present three such cases. When discussing Harel and Sowder (2003), we show how a student's penchant for empirical arguments does not imply that she believes empirical arguments can provide certainty. When discussing Morris (2002), we show how a student desiring more evidence of a claim after reading a deductive argument that supports it does not imply that this student believes that deductive arguments do not provide certainty. Finally, looking at Weber (2010), we illustrate how providing students with the space to make more nuanced evaluations of arguments can reveal the sophistication of their understanding of justification and proof.

Evaluations of an empirical argument

Sowder and Harel (2003) described an undergraduate mathematics major, Ann, who struggles with proof. Sowder and Harel presented a transcript where "Ann showed evidence, as many students do, of relying on the inductive proof scheme (based on examples)" (p. 256). Ann was presented with the claim that $(A^T)^{-1} = (A^{-1})^T$. She was then shown that the claim held with a specific two-by-two matrix by calculation. Ann was asked if she thought this would always be the case.

Ann: … *I think* it is. *I think* it is true.

Interviewer: How would you convince me, or convince somebody else, that this was true?

Ann: *I guess* you would just have to go through all of these, or go through a couple of other ones, or *maybe deal with letters so you can see that … if the letters work then you can just plug in anything for the letters.*

Interviewer: Which is *more convincing,* doing some more examples or doing it with letters?

Ann: To me personally, I think doing more examples of it. But if you show it can work for anything, *that's even more convincing.* But for me, I like to see the numbers, and be able to see that. Yes! It works for this case, and this case, and this case. (Sowder & Harel, p. 256, emphasis added)

We see strong evidence that Ann had absolute conviction neither in the claim $(A^T)^{-1} = (A^{-1})^T$ nor the notion that empirical arguments provide absolute conviction. When asked if $(A^T)^{-1} = (A^{-1})^T$ would always be the case, Ann qualified her response with the phrase "I think" (as opposed to "I know" or "It is definitely true"), usually used to indicate the retention of some doubt. When asked how she would convince others, Ann expressed hesitation ("I guess"). In addition to checking more individual examples, Ann also suggested an algebraic demonstration because of its generality. When asked which approach is better, Ann expressed a preference for constructing an empirical argument but indicated an algebraic proof would be *"even more convincing"*. The point here is *not* that Ann actually holds a deductive proof scheme (Sowder and Harel attributed her comment about "using letters" to a ritualistic awareness that variables can play a role in a proof). Rather, the point is that since Ann was aware that some arguments can be *"even more convincing"* than empirical ones, she cannot be claiming that empirical arguments provide absolute conviction. Ann's personal preference for empirical arguments might be troubling to a mathematics educator, but her penchant for examples does not contradict any of our core assumptions about mathematical practice.

Students are not convinced of a statement after reading its proof

Morris (2002) conducted a study in which she presented two deductive arguments and two empirical arguments in support of number theory statements to 30 pre-service teachers. Nine of the pre-service teachers thought that neither the empirical arguments nor the deductive arguments were sufficient to establish the statements, desiring further evidence in support of the statements. Morris questioned whether these students appreciated the generality and conclusiveness of a proof. She presented the following representative explanation that one student provided for his or her justification.

P: You put them [the two deductive arguments] together because since it's more of a mathematical proof, it's *probably almost certainly true.* There's *less chance for a mistake* or exception in this pile [the two deductive arguments] than there is in this pile [the two empirical arguments]. (Morris, 2002, p. 96, emphasis added).

In this excerpt, P preferred the deductive arguments to the empirical ones, yet nonetheless retained some small doubt ("probably almost certainly true") about the truth of the statement. However, P's retention of doubt is not necessarily non-mathematical if the doubt is because the participant lacks absolute conviction in the correctness of the proof. By saying "there's less chance for a mistake" in the deductive arguments (rather than no chance of a mistake), P appears to be saying that he retains some doubt that the deductive arguments are correct. Hence we judge P's rationale to be consistent with Paseau (2011) who said "that we are in possession of a proof of p does not mean we should be certain of p". Indeed, Morris (2002) raised this possibility earlier in her paper: "It should be noted that even deductively derived mathematical conclusions are not absolutely certain. Mathematical knowledge is not infallible [...] There have been incorrect proofs" (p. 87).

The effect of binary choices on conviction

The following excerpt comes from a study by Weber (2010). He asked 28 undergraduate mathematics students to evaluate ten arguments on how convincing they found the argument on a five-point scale, with five meaning "I am completely convinced by this argument", and on whether the argument was (a) a fully rigorous proof, (b) a proof that was not fully rigorous, or (c) an argument that did not meet the standards of proof. Weber found 39 instances in which a participant claimed the argument was a proof but did not find the argument to be completely convincing (*i.e.*, gave the argument a conviction score of 4 or less). At first glance, this finding would seem to imply that students thought proofs were not definitive. However, in 23 of the 39 instances, the participants claimed they made their judgments because they thought the proof was valid but were not certain. For instance, one participant made the following comment after reading a proof that "$4x^3 - x^4 = 30$ has no solutions" based on showing that the maximum value of $f(x)=4x^3 - x^4$ was 27.

P19: For convincing, hmm, I'll give it a 4. Is it a proof? Yes they've shown it [$f(x)$] has a maximum of 27 and is decreasing after that so it could never be equal to 30. Yes, it's a rigorous proof.

Reading only this excerpt, one might infer that P19 did not appreciate that proofs should provide absolute conviction. However, the follow-up prompt by the interviewer revealed this was not the case.

Interviewer: Okay. Even though you can't find anything wrong with it, you just don't find it convincing?

P19: I have to think about this one [...] *I think* it's right, the proof. But *I don't know.* Let me just put I consider the argument to be a proof although not fully rigorous. (Weber, 2010, p. 327, emphasis added).

When P19 judged the argument to be a proof, he had relative conviction that the deductive argument he read was valid ("I think it's right, but I don't know"). When queried, he changed his judgment about the rigor of the proof. P19 initially evaluated the argument as a rigorous proof, but then lowered his evaluation to a proof that was not fully rigorous. We did not think the proof lacked rigor. (Interested readers can read Proof 4 in the appendix of Weber, 2010, and form their own conclusions). P19 did not cite any feature of the proof that was not rigorous. Although we do not know how P19 interpreted the word "rigorous", we conjecture that P19 did not want to state whether the argument was definitely a proof. To avoid this choice, he interpreted a non-rigorous proof as meaning a proof that was *probably* correct.

Implications

Our analysis offers methodological, theoretical, and pedagogical implications. Methodologically, claims that students hold non-normative views about conviction, justification, and proof are usually based on the inference that students have absolute conviction of some type: (i) students have absolute conviction in a mathematical statement on the basis of empirical evidence, or (ii) they are absolutely convinced that a deductive argument in support of a statement is valid yet still have doubts about the statement. Regarding (i), our analysis reinforces the recommendation of Inglis and Mejía-Ramos (2008) that researchers need to actively verify that students have absolute conviction (not merely relative conviction) in the empirical arguments they produce or evaluate. Regarding (ii), we believe it is *not* necessarily problematic if students harbor doubt in a statement after reading or producing a proof of the statement. For this situation to be problematic, the researcher would need to verify that these students had absolute conviction that the proof was correct, an occurrence we expect to be unusual. As Fischbein (1982) noted, arguments that bestow absolute conviction are unique to mathematics so we should not be surprised that students do not often obtain it.

In clinical interviews, researchers can strengthen their inferences with a follow-up question: "Are you absolutely certain that this claim is true?" or "Do you have any doubts at all that this proof is correct?" If using survey-type items, researchers can ask students to rate how confident they are in their evaluations using a five-point scale, with a 5 being "I am absolutely certain that this statement is true" or "I am absolutely certain that this proof is correct"[6]. Interpreting students' cognition with the use of a paper-and-pencil survey is always problematic; in this case, we will be unsure what students mean by "absolutely certain". Nonetheless, this approach at least provides the student with the space to express doubt about his or her judgment. The guiding principle is that the researcher should explicitly ask the participants if they are speaking of absolute certainty and explicitly provide participants with the opportunity to qualify their responses.

If a researcher interprets a student's level of conviction as representing a subjective probability, this subjective probability is likely affected by a multitude of factors, such as the student's mood and the relationship between the student and the teacher or researcher. These factors were not described in the excerpts discussed above, largely due to the fact that these issues were not discussed in the papers in which the excerpts originally appeared. Such factors would be useful to consider when analyzing students' utterances in the future.

Theoretically, we doubt that many students obtain absolute conviction in any of the new mathematics that they are learning. It is indeed troubling that students will submit empirical arguments or prefer empirical arguments to deductive ones. However, we find that the explanation that students do so because they gain absolute conviction from empirical arguments is inadequate as they rarely obtain absolute conviction on the basis of any mathematical argument. Instead, students' propensity to produce or accept empirical arguments might be based on many factors. In comparison to empirical arguments, deductive arguments are harder to produce, understand, and verify. Students may lack the interest or confidence to try to write or understand a proof. Exploring the plausibility of these hypotheses and considering other factors may provide a more nuanced account of students' mathematical behavior with respect to justification and proof.

Pedagogically, our perspective problematizes why we might introduce proof to students. A typical rationale amongst K-12 teachers is that the reason for proving statements in the mathematics classroom is to convince students that the statements are true (*e.g.*, Knuth, 2002), but we argue that proof may be unable to play that role for students. Nonetheless, we think proof still may play a valuable role in the classroom, in that proof can satisfy other epistemic aims (Weber, Inglis & Mejía-Ramos, 2014), such as offering explanation or insight and introducing students to the social practices of mathematicians. We believe that our perspective offers teachers both more flexibility and more responsibility. With flexibility, we contend that if the goal of an argument is for students to obtain conviction in a statement, the teacher should feel free to use other types of non-deductive arguments, since such arguments may be more persuasive to students than proof. With responsibility, teachers still may want students to interact with proof, both because it is central to the practice of mathematics and for the potential insights proof might provide. When teachers do so, they have the responsibility to think carefully about what epistemological benefits that proof can have for students, since it is not enough to simply claim that proving is *prima facie* necessary or useful because it is the best means to obtain conviction.

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Notes

[1] We believe there is substantial variation in what mathematicians may accept as a logically necessary consequence. For instance, mathematicians may disagree about how large a gap is permissible in a proof, whether a computer may aid in making such a deduction, or whether an inference may be drawn from a diagram. For a further discussion, see Dreyfus (2004).

[2] We do not attempt to give a complete review of this research; such reviews can be found in Reid and Knipping (2010) and Harel and Sowder (1998). We have cited representative studies to illustrate the type of work that is done in mathematics education.

[3] For a more detailed account of Hales's proof, see Devlin (2003).

[4] Doyle, A. and Johnson, S., "Scientists more convinced mankind is causing global warming", *Reuters*, 27 September 2013, retrieved from www.reuters.com/article/2013/09/27/us-climate-ipcc-idUSBRE98L0KD20130927

[5] Regarding the first point, we suggest that mathematicians may have absolute certainty in the deductive method's ability to transmit truth, even if they may lack an epistemic warrant for doing so.

[6] KW is thankful to Eric Knuth for this methodological heuristic. Knuth suggested that I include this five-point scale when I was designing the study reported in Weber (2010).

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[…] neither reading books nor working exercises,—whether they be trivial or serious,—will suffice to develop the reasoning powers. An analytical method of procedure is requisite which shall perfect one by one our performance of the different mental operations that enter into the business of inquiry. Now the mental operations concerned in reasoning are three. The first is Observation; the second is Experimentation; and the third is Habituation.

Peirce, C. S. (1992) *Reasoning and the Logic of Things* (ed. Ketner, K. L.), p. 182. Cambridge, MA: Harvard University Press.

Apologies to Gaya Jayakody and Rina Zazkis for two unfortunate errors that crept into their article in FLM 35(1). First, on page 9, the second part of Definition D2 should read:

And f is discontinuous at $x = c$ in its domain if,

$$\lim_{x \to c} f(x) \neq f(c)$$

And on page 11, the function shown in Figure 3 should, of course, be:

$$f(x) = \frac{x^2(x-3)}{(x-3)}$$