# SEAN *vs.* CANTOR: USING MATHEMATICAL KNOWLEDGE IN 'EXPERIENCE OF DISTURBANCE'

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"The use of mathematical knowledge in teaching is often taken for granted" (Ball & Bass, 2000, p. 86). Only a small number of experiences of such usage remain memorable. They begin with what Mason (2002) refers to as 'disturbance':

> Most frequently there is some form of disturbance which starts things off. It may be a surprise remark in a lesson, […] or a moment of insight (p. 10).

The story 'Sean vs. Cantor' presents two such experiences of disturbance that we analyse from mathematical and pedagogical perspectives.

## Background and setting

Our story is situated in the course 'Foundations of Mathematics', a Master's course for practicing secondary mathematics teachers. The main character is Lora, an experienced instructor who has taught several offerings of this course. 'Foundations of Mathematics' introduces students to several fundamental ideas and 'big theorems' in mathematics, which either were long forgotten or were not encountered in students' undergraduate studies. Infinity and Cantor's method of corresponding infinite sets were among the topics explored in the course.

The idea of infinity was introduced in a friendly and 'playful' manner via the exploration of famous paradoxes, such as Hilbert's Hotel Infinity and the Ping-Pong Ball Conundrum (Mamolo & Zazkis, 2008), before introducing students to the conventional mathematical understanding of the presented ideas. The discord between intuitions and formal mathematics – such as reasonable intuitive beliefs that the set of even numbers is smaller than the set of natural numbers, or that the set of natural numbers is smaller than the set of rational numbers – was explicitly acknowledged. However, this practical intuition, based on the reasoning of inclusion appropriate for finite sets, is inconsistent with the normative standard for comparing infinite sets. Formally, these sets are considered equinumerous, all having the same cardinality, denoted by $\aleph_0$. The proofs are based on establishing a one-to-one correspondence between the pairs of sets. For the latter case, setting up a one-to-one correspondence relies on a systematic listing of rational numbers so that they are all guaranteed to appear on the list, as illustrated in figure 1.

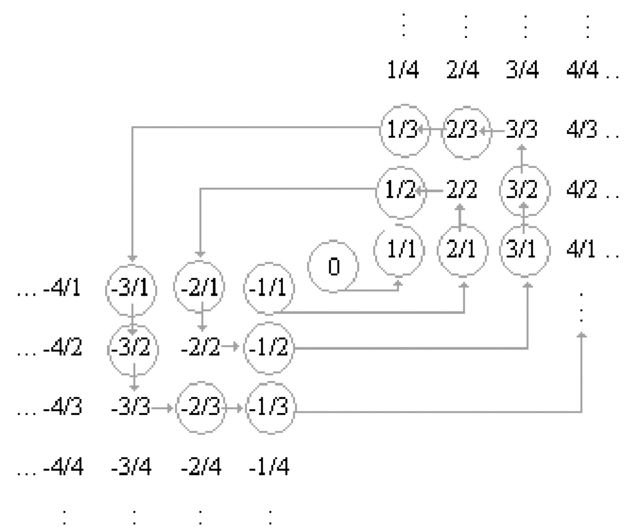


*Figure 1. Corresponding rational and natural numbers*

Unwinding the spiral produces a list in which each rational number is paired with exactly one natural number (0→1, 1/1→2, -1/1→3, 2/1→4, etc.), and as such demonstrates that the cardinality of the two sets is the same.

Having explored a variety of sets with cardinality equal to $\aleph_0$, students were introduced to Cantor's theorem, that the cardinality of the set of real numbers is greater than $\aleph_0$, or, in informal terms, that there are more real numbers than natural numbers. The proof of this theorem is often referred to as Cantor's diagonal method (see, *e.g.*, Burger & Starbird, 2000, for a proof).

## Sean's correspondence

Following a discussion of different infinities, Sean suggested a correspondence between real and natural numbers that he argued was one-to-one. (In fact, Sean's suggestion attends only to real numbers in the interval (0,1), but so does the conventional presentation of Cantor's proof of the theorem in question.) Sean suggested considering numbers in their decimal representation, as follows:

> Start with numbers that have only one (non-zero) digit after the decimal point, and correspond them to the first nine natural numbers: 0.1→1, 0.2→2, … 0.9→9. Then look at numbers with 2 digits after the decimal point, avoiding those with 0 at the end, and correspond them to the natural numbers from 10 to 99:

> $0.01 \rightarrow 10$, $0.02 \rightarrow 11$, … , $0.09 \rightarrow 19$, $0.11 \rightarrow 20$, … , $0.99 \rightarrow 99$
>
> Then take all the numbers with 3 digits after the decimal point, avoiding those with 0 or 00 at the end, and correspond them to the next 'bunch' of natural numbers, and so on. This method presents an 'ordering' of real numbers, and so their cardinality is $\aleph_0$.

This suggestion of Sean's created an initial 'experience of disturbance' for Lora. She presented various arguments to refute Sean's claim, however he resisted accepting them. Sean's persistence directed Lora to seek other, more convincing, refutations.

The fact that it is impossible to find a natural number that corresponds, for example, to $1/3$ (as it has infinite non-zero decimal representation) did not convince Sean. He insisted that when corresponding the natural numbers and rational numbers in class, an explicit 'match' for each fraction was not specified, but instead the emphasis was on the *possibility* of listing the rational numbers 'in order'. Sean claimed that his method presented a possible ordering, that "*eventually* will get to $1/3$ or any other number". There was an apparent confusion in his reasoning – alerted to in previous research (*e.g.,* Mamolo & Zazkis, 2008) – between a very large number of digits after the decimal point, and an infinite decimal representation.

What finally convinced Sean was Lora's observation that when the rational numbers were 'ordered' by unwinding the spiral it was possible to determine what came before and after each number. Lora asked Sean to determine what real number was placed before or after $1/3$ in his ordering. Failing to determine this convinced Sean of the inappropriateness of his correspondence. However, Lora was unhappy with her approach of immediately refuting a student's suggestion, rather than giving him, and others, an opportunity to explore it in greater detail.

## Creating a conflict

It was a year later teaching the same course that Lora presented 'Sean's correspondence' to a new group of students. Following the class' exposure to Cantor's theorem and proof, Lora introduced for the students' scrutiny Sean's suggestion for a correspondence between sets of real and natural numbers. Her goal was to provoke a cognitive conflict in her classroom, a goal motivated by the abundant research suggesting that this is a powerful pedagogical approach (*e.g.,* Tirosh & Graeber, 1990). However, it is only after students realise the existence of a conflict, a potential conflict created or recognised by the instructor becomes a cognitive conflict for students (Zazkis & Chernoff, 2007). To Lora's surprise, the conflict – presented by the fact that Sean's correspondence contradicted the established proof of Cantor's theorem – was not immediately recognised.

"Cool!" – was the reaction voiced by one of the students, and several nods around the table indicated other students' agreement. Lora was extremely surprised with this reaction. She expected immediate recognition that something was wrong with Sean's suggestion. She further expected that such recognition would be followed by a search for a flaw in his argument, and this might require some time and 'scaffolding'. What initially appeared as mutual acceptance of the argument was totally unexpected; it contradicted not only the specific proof of Cantor's, but also the essence of a 'mathematical theorem'. This was yet another 'experience of disturbance' for Lora. So both confused and 'disturbed' Lora did what she found, as most teachers, very difficult to do: she said nothing and waited.

## Conflict recognition

As implied earlier, the first step toward conflict resolution is conflict recognition. This step was made by a very quiet remark of one student: "So are you saying that Cantor was wrong and Sean should get a Fields medal?"

Yet again, Lora said nothing, restraining a smile. The next set of claims came simultaneously from different corners of the class:

- "Wait, there must be something wrong."
- "It is more likely that Sean is wrong rather than Cantor."
- "We must be missing something. It is unlikely that no one thought of disproving this for the last two hundred years."

## Conflict resolution

Once the conflict was acknowledged, the flaw in Sean's argument was not difficult to detect. The first realisation towards refuting the argument was that the real number $\pi$ – a generic example for an irrational number – does not have a match within natural numbers, as its decimal expansion is infinite. However, since $\pi$ is not included in the interval (0,1), this observation was modified, recognising that none of the irrational numbers between 0 and 1 would have a 'natural match' in Sean's correspondence because of their infinite decimal expansion. While this was sufficient to conclude, in a student's semi-cynical words: "Sean was wrong, Cantor was right, surprise, surprise", Lora sought a more detailed 'resolution'.

## If not, what yes?

Following consensus on the refutation of Sean's method, Lora posed the following question: "So we agree that Sean's correspondence does not prove that real numbers and natural numbers have the same cardinality. But what does it prove, if anything?"

This question was inspired by Koichu's (2008) 'If not, what yes?' extension of the famous pedagogical strategy 'What if not', described by Brown and Walter (1993). Koichu's approach is based on presenting students with a mathematical claim that has to be refuted, and then asking, 'Since this statement is wrong, which one would be correct?' inviting gradual modifications of the presented claim. In our case, if Sean's correspondence does not demonstrate what was initially intended, what (if anything) *does* it prove?

The initial suggestion to 'what yes?' was that Sean's correspondence presented a matching of rational numbers with natural numbers. Some students considered this as a 'simpler' correspondence than the 'conventional' one introduced in class. However, once again, the 'wisdom of Sean' was called into question. The sceptical voice suggested that if it were that simple, then why would textbooks introduce the spiral method?

Eventually, the students realised that only a subset of the set of rational numbers was included in Sean's correspondence – namely, the set whose elements have a finite decimal representation. Thus, the correspondence that Sean created proves that the set of rational numbers with a finite decimal representation (in the interval (0,1) ) has the same cardinality as the set of natural numbers. While finding yet another set with cardinality $\aleph_0$ was not a very exciting mathematical conclusion, it was a worthwhile mathematical engagement for students.

## Mathematics in teaching: examining complexities

Growing attention in mathematics education research to teachers' knowledge has led to revisiting and refining the classical categories of subject matter knowledge and pedagogical content knowledge (Shulman, 1986). On one hand there are attempts to 'zoom out' and consider more general constructs, such as *knowledge of mathematics for teaching* (Ball & Bass, 2000) or simply *mathematics-for-teaching* (Davis & Simmt, 2006). On the other hand there are attempts to 'zoom in' and refine the deeply intertwined notions of mathematics and mathematical pedagogy. This results, for example, in a more detailed examination of the relationship among mathematical and pedagogical goals (Liljedahl, Chernoff, & Zazkis, 2007) and in introducing additional sub-categories of teachers' knowledge (Hill, Ball, & Schilling, 2008). Such refinements sharpen the lens through which the complexities of mathematics in teaching can be viewed and analysed.

## Mathematics in teaching: complexity of using

Liljedahl *et al.* (2007) illustrated a way of examining the use of tasks in teacher education with a 2 × 2 array, presented in figure 2. They suggested reading the content of the four cells as *"The use of **x** to promote understanding of **Y**"*. The array disaggregates the "knowledge of mathematics and use of pedagogy from the mathematical and pedagogical understandings we wish to instil within our students" (p. 240).

Though this framework was developed for the analysis of task design in teacher education, it serves well in analysing the classroom experiences reported above from the perspective of each cell.

### mM: The use of mathematics to promote understanding of Mathematics

Lora's original goal was to demonstrate the difference in cardinality between the sets of real and natural numbers. Lora's deep understanding of mathematics equipped her with tools for refuting Sean's argument. However, it was the creation of a 'convincing argument for Sean' that exemplified the usage of mathematical knowledge in teaching that enhanced students' understanding of mathematics. It was the search for additional arguments, building upon the claims presented by Sean regarding the issue of 'ordering' numbers as the basis for the argument of correspondence, that turned Lora's mathematics to Sean's eventual mathematical understanding.

| | | GOALS | |
|---|---|---|---|
| | | Mathematics (M) | Pedagogy (P) |
| USAGE | mathematics (m) | **mM** | **mP** |
| | pedagogy (p) | **pM** | **pP** |

*Figure 2. Goals and usage grid*

When Lora revisited Sean's argument with a new group of students, it was again her sophisticated mathematical understanding which enabled her to engage students effectively in the 'if not, what yes' investigation, and to develop their own mathematical understanding of the relevance and scope of Sean's argument.

### pM: The use of pedagogy to promote understanding of Mathematics

Liljedahl *et al.* (2007) recognise that teachers "need not only to be aware of the mathematics embedded within the task, but [also] … need an understanding of how to mobilize this knowledge for their students' learning" (p. 240). This awareness and understanding of Lora's resulted in her developing two different pedagogical approaches: creation of cognitive conflict and 'if not, what yes' investigation. Through both of these approaches, students developed specific mathematical knowledge regarding properties of infinite sets, in particular the sets of real and rational numbers. Further, Lora's usage of pedagogy promoted students' understanding of mathematics more broadly. After Lora's surprising experience regarding students' initial acceptance of Sean's argument despite the conflict with established mathematical knowledge, a new mathematical goal developed – one which related to the meaning of a mathematical theorem. This goal was met through students' experiences and resolutions of cognitive conflict, and through their own mathematical endeavours to refine the claim associated with Sean's argument.

### mP: The use of mathematics to promote understanding of Pedagogy

Though the tasks themselves did not have explicit pedagogical goals, these goals are always present when working with teachers. Creating a cognitive conflict can be seen as exemplification of general pedagogy. It was appreciated by students, who are practicing teachers, as a powerful strategy for some of their future endeavours. Further, in resonance with Koichu's (2008) encounters, students also recognized the 'if not, what yes' pedagogical approach as "an experience in 'genuine' doing of mathematics" (p. 450), and as such saw it as a compelling pedagogical tool.

### pP: The use of pedagogy to promote understanding of Pedagogy

While explicit pedagogical moves have been acknowledged earlier, there is a simple one that may not be noticed without drawing attention to it. This is Lora's move of remaining silent, saying nothing. If used only occasionally, it serves as a powerful tool in promoting exchange of mathematical ideas among students. In working with teachers, it exemplifies a very simple technique, and experiencing its effects invites teachers to try this in their own classroom.

## Mathematics in teaching: complexity of knowing

We introduced and analysed two examples of 'experience of disturbance' in the described events. The first was an unusual idea presented by Sean, suggesting an inappropriate correspondence between the sets of real and natural numbers. The second was an unexpected first reaction from a class of mathematics teachers, initially accepting Sean's correspondence. In both cases the disturbance resulted in a positive twist: it triggered further 'unpacking' and expansion of Lora's knowledge into its new forms.

The first disturbance enriched Lora's understanding of students' potential difficulties and triggered the search for and development of additional refuting arguments, where the explanatory power of different arguments – though equivalent mathematically – was perceived differently by a student. Using the terms developed by Hill, Ball, and Schilling (2008), this disturbance helped Lora acquire a more profound *knowledge of content and students* (KCS) as it enriched her repertoire of possible incorrect ideas and unhelpful intuitions that may be held by students. In particular, it enriched Lora's "knowledge of how students think about, know or learn this particular content" (*ibid.,* p. 375). As a result, Lora's *common content knowledge* (CCK) and her *specialized content knowledge* (SCK) developed into *knowledge of content and teaching* (KCT), as evidenced in the enhanced variety of explanations that she developed. This disturbance also resulted in developing tasks for another group of students, implementing the pedagogy of 'cognitive conflict' and 'if not what yes', which can be seen as further indicators of extending Lora's KCT.

The second disturbance resulted in a new understanding of the mathematical dispositions of a group of practicing teachers. The fact that students' reactions were unexpected, but dealt with skilfully, is a clear sign of Lora's enhancement of her KCS. It also shows the strong interrelationship between KCS and KCT, as deeper understanding of the former contributes to growth in the latter.

## Conclusion

We agree with Ball and Bass (2000) that "[n]o repertoire of pedagogical content knowledge, no matter how extensive, can adequately anticipate what it is that students may think, how some topic may evolve in a class, the need for a new representation or explanation for a familiar topic" (p. 88) and that "[b]eing able to use mathematical knowledge involves using mathematical understanding and sensibility to reason about subtle pedagogical questions" (p. 99).

The story of 'Sean vs. Cantor' illustrates exactly this: unanticipated students' thinking and the teacher's skill and sensibility in developing new explanations and new instructional engagements. Our main contribution is in exemplifying how ideas developed and adopted by studying the work of elementary school teachers are transferable to teaching undergraduate mathematics. Lora's experience further demonstrates the importance of 'noticing' and 'disturbance' (Mason, 2002) in transferring personal *knowing* of mathematics into *using* it in teaching.

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