CHOICE AND USE OF EXAMPLES AS A WINDOW ON MATHEMATICAL KNOWLEDGE FOR TEACHING

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Teachers regularly use examples in mathematics lessons. Purposes include demonstrating how to solve typical problems, creating cognitive conflict, providing practice for students through assigning exercises, and using examples to set the scene for further learning experiences.

Although the term *example* is probably understood it is helpful to define it: "a specific instantiation of a general principle, chosen in order to illustrate or explore that principle" (Chick, 2007, p. 5). This definition highlights the teaching and learning purpose of an example: it is not the specific situation or answer that matters most for learning, but rather the general principles illuminated by the example. Thus, for instance, it is not that 41 has especial interest as a prime number, but that it can demonstrate the meaning of primeness (and, additionally and usefully, goes beyond the all-too-well-known "small" primes).

An overview of the history and role of examples has been given by Bills, Dreyfus, Mason, Tsamir, Watson, and Zaslavsky (2006). Our concern is with how to choose examples that allow the general principles to be noticed above the "noise" of the specific details (see Skemp, 1971, pp. 29-30). This requires of the teacher an awareness of what an example has to offer. The term *potential affordances* - derived from Gibson (1977) - refers to the opportunities inherent in a task or example, whether perceived or not. (*Affordance,* as defined by Gibson, referred only to what the user perceives. It is useful, however, to acknowledge that there may be *other* opportunities offered by an example. Important teaching opportunities may be lost if a teacher overlooks what an example offers.) Recognising these affordances, ensuring they are present in an example, and bringing them out when the example is used in the classroom requires a deep understanding of mathematics, but with a teaching/learning focus, since the example's purpose is educative.

An investigation of how teachers choose and use examples in the classroom should thus provide insights into their knowledge of mathematics for teaching. Moreover, this area is worth examining because examples are at the critical nexus between pedagogy and content: the example has to demonstrate (pedagogy) a general mathematical principle (content) despite its specific status. As a result, a study might resolve, or at least question, some of the constructs included in "mathematical knowledge for teaching" (see, for example, Hill, Sleep, Lewis, & Ball, 2007). To explore this, I examine the work of two Australian Year 6 primary (elementary) teachers, both teaching two lessons on ratio. By considering the examples that they used, we gain insight into what mathematical knowledge they had about ratio and whether any aspects of that mathematical knowledge were specific to the teaching context. I also suggest that one important outcome of such a study is to reveal what specific aspects of mathematical knowledge for teaching are important, over and above a concern about whether or not any particular teacher possesses such knowledge.

The teachers

Clare had trained as a primary teacher through a one-year graduate program and had five years' classroom experience. Hilary had been a secondary science teacher, but changed to primary teaching and was in her second year of teaching at that level. They were involved in a larger study investigating pedagogical content knowledge and could nominate which lessons they were happy to have studied. Each taught two ratio lessons that were videotaped; these were the first lessons on ratio for their classes. This exploration of the examples used as they taught is intended to highlight the mathematical knowledge needed for teaching ratio, and how that is revealed through the choice and use of examples. A more detailed analysis of the lessons was presented by Chick and Harris (2007).

Introducing ratio

Both teachers' lessons had an introductory component. Hilary began by explaining, initially without giving a definition or example, that ratio is associated with fractions or proportion, and is used to show the amounts that comprise a whole. She then used ten students to illustrate the ratios 3:7 (light hair to dark hair) and 5:5 (boys to girls). She wrote formal notes about ratio on the board for students to copy, defining ratio as a comparison of two or more related numbers. Her definition included an example, 1:5, without a physical context, and her notes suggested this could also be written as $^1/_5$. She later illustrated the 1:5 example in the context of making cordial, highlighting that one part of cordial and five parts of water should be used, to give a total of six equal parts. She clarified that the actual size or amount of these equal parts did not matter, provided all parts are equal, and emphasised the order of the numbers, thus indicating understanding of some key components for ratio; however, she did not address the connection to the fraction $^1/_5$.

For the Learning of Mathematics **29**, 3 (November, 2009)
FLM Publishing Association, Edmonton, Alberta, Canada

In contrast, Clare did not mention ratio at all during the first 25 minutes of her lesson. Instead she had students colour 2 cm × 3 cm rectangles so that for every square coloured red, two squares were coloured blue. When she asked how many squares were coloured red and how many were blue, one student pointed out that $^1/_3$ of the rectangle was red and $^2/_3$ was blue. This allowed Clare to explore the connection between fractions and the situation that they had, highlighting that the $^1/_3$ came from the fact that $^2/_6$ of the squares in the rectangle had been coloured red. Students were later permitted to colour half squares, and Clare asked students to find all the different colourings of the 2 cm × 3 cm rectangle using the "one red for every two blue" scheme. As students worked there was discussion about equivalent arrangements, symmetry, and how to work systematically. At the end of this exploration of a single example Clare finally defined ratio, using the 2 red to 4 blue example. She identified the relationship between the parts and the whole, made the connection to fractions correctly, had students simplify the ratio 2:4 to the "basic" ratio 1:2, and linked this back to her original instruction to colour 1 red and 2 blue. Clare then used the remainder of the lesson on two additional situations. Students worked with a 5 cm × 3 cm rectangle, colouring two squares red for every three blue. They correctly responded that this meant a ratio of 2:3, and although 6:9 was the only resulting equivalent ratio, it was shown in many different arrangements since part of Clare's emphasis was on systematically enumerating the possibilities. Clare also explained that the fraction for the proportion of reds involved is $^6/_{15}$ and not $^6/_9$, by highlighting what constituted the whole. Clare concluded her first lesson with another extended task example, having students find rectangles that could be coloured in the ratio of 1:3, with one red to three blue, using whole squares only. Clare helped the students recognise that they needed to search for rectangles with area divisible by 4; this allowed students to find more equivalent ratios and see the role of the total of the two parts in the ratio. Class discussion then highlighted the equivalent ratios 1:3, 2:6, 3:9, and so on, and one student observed that finding a number that goes into both parts of the ratio will allow simplification of the ratio. This example afforded the development of many important ideas for ratio, even though it considered only one ratio value.

In Hilary's case, she gave two numerical examples in a discrete situation (number of students), followed by a different example in a continuous context (the cordial mixing). Her 5:5 example of girls and boys arose as an accidental consequence of the group of students selected, and she did not capitalise on the affordance this gave to simplify this ratio or make sense of the idea of simplification. This was in contrast to Clare who – while also addressing other mathematical topics like area, combinatorics, and symmetry – had chosen her first example in such a way that it was natural to talk about the simplified ratio. Despite the fact that ratio was not mentioned until well into the lesson, the rectangle examples, with their continuous context, provided Clare's students with the opportunity to make clear connections to fractions, parts, wholes, and simplification. Hilary, on the other hand, certainly recognised that fractions and ratios are inextricably linked, but she did not distinguish between the ways parts and wholes are used in the two contexts. Moreover, no work was done at this stage on the simplification/equivalence issue, despite the opportunities afforded by one of the examples.

Further development of ratio ideas

Both teachers conducted similar practical activities, Hilary after her introduction, and Clare at the beginning of her second lesson. In Hilary's class each student pair received a bundle of straws, and determined the ratio of red straws to blue straws, emphasising the order, and simplifying the ratio obtained. Hilary's instructions suggested only that simplification for ratios involved doing the same kind of simplification that students were used to doing for fractions. The chosen examples were 5:10, 12:4, 8:6 and 15:3, all of which simplify readily. Groups then combined their straw bundles and obtained ratios for the new groupings, simplifying if possible. Hilary appeared to anticipate that some would not simplify.

Clare's activity involved several containers holding two groups of coloured blocks, and she required students to count the blocks, determine the ratio, and then simplify to get the basic ratio, for as many containers as possible. The chosen examples were 21:14, 32:8, 20:40, 15:6, 21:7, 15:25, 10:15, 6:18, 14:8, and 20:25. When discussing the answers there was good attention to the operations involved in simplifying, but less care was taken in maintaining the order of the parts. For example, 1:4 was accepted as the basic ratio for 32:8, perhaps because some students recorded the original ratio as 8:32 since the order was arbitrary.

It is interesting to note what mathematical issues can and cannot be afforded by the chosen examples. Neither teacher included a 1:1 situation, and although Clare had examples that might simplify to the same ratio (unlike Hilary), she did not address this, making this a potential but unutilised affordance. Hilary focused on notation and order, but although she incorporated simplification there was no discussion of the conceptual connection between the simplified ratio and the non-simplified one. Clare's activity structure did not emphasise order, but students had more practice with simplification and the operations required.

Dealing more formally with equivalent ratios

The operations to simplify or produce equivalent ratios were not made as explicit in Hilary's lessons as they were in Clare's. After reviewing simplification at the beginning of the second lesson, Hilary used the example of 5:20 to illustrate that doubling both numbers, just as in fractions, produces an equivalent ratio. When she asked students to suggest how to produce an equivalent ratio for 9:12 the first suggestion was to again double the numbers. When Hilary asked for a second answer not involving doubling, one student suggested multiplying by ten. Hilary said any number could be used for multiplying but did not use an additional example to show this. Although doubling and multiplying by ten are mathematically correct methods, the general principles of equivalence were obscured: there was no emphasis on the fact that doubling means multiplication by two, and, for the multiplying by ten case, Hilary obtained and explained the new numbers by "adding a zero on the end", again hiding the multiplication operation.

Later Hilary asked students to find equivalent ratios for 1:10, 13:13, 2:3, and 6:9:12. When a student sought help producing an equivalent ratio for 13:13, Hilary asked if it simplified, clarifying this by asking if anything "goes into 13". Hilary appeared too mindful that 13 is prime, because after accepting the student's response that nothing seems to go into 13, she had students construct an equivalent ratio involving larger numbers instead of simplifying to 1:1. Hilary may not have had a complete understanding of the definition of prime number (having two distinct factors); perhaps she recognised the primeness of 13 but overlooked the two special factors because of their special nature. Interestingly, in a later class discussion, one student proposed 2:2 as an equivalent ratio for 13:13, explaining that he had reduced it to 1:1 and then doubled. Hilary accepted this explanation, adding that if you have half the people in one group and half the people in the other group, then it doesn't matter how many people are in the groups. She then wrote down 1:1 as the equivalent ratio, but explained that this was permissible even though the two did not go into 13 and they had not been able to simplify by finding a number that went into both 13s. On another occasion a student asked about simplifying the ratio 2:3 by dividing by two. Hilary did not permit the possibility of expressing the ratio as 1:1.5 because, in Hilary's words, "two does not go into three evenly". In both these instances, Hilary's understanding about primeness and the divisibility of three by two was incompletely applied in the ratio context. In the latter case, she seemed aware of the convention of simplest form having only whole numbers, but unaware that unitary comparisons are actually important in "rates" relationship.

Clare made the operations for equivalent ratios more explicit. She revisited the rectangle colouring example, recording 2:4 and eliciting the simpler ratio 1:2 from students, and writing it underneath. As students described how to change one ratio into the other, she drew arrows from the 2 and 4 of the first ratio to the 1 and 2 respectively in the second. She explicitly recorded "÷ 2" next to each of the two arrows, and made the connection to equivalent fractions. Clare also highlighted that the basic ratio 1:2 did not tell them how many squares had been involved in the original situation. She set up a table in which she made explicit connections between the parts, the total, and equivalent ratios. Still using the red/blue colouring in rectangles as the context, she headed the table with "Red", "Blue", and "Total". Using a ratio of 2:3, she recorded 2 and 3 in the Red and Blue columns respectively, and put 5 in the Total column. She asked students to consider what would happen if she now had a rectangle with 15 squares, and wrote 15 in the Total column. She asked what had to be done to the 5 to turn it into 15, and when students responded "× 3", she wrote this beside an arrow joining 5 and 15, and then elicited from the students that the same operation needed to be applied to the parts to get the equivalent ratio 6:9. She then worked through a "simplify" example, set up using the same table structure, and emphasised the role of the common factor. Prior to students tackling some worksheet problems Clare used this three-column table approach, with emphasis on the multiplying factor, to work through different cases of a real-world situation: "Chris and Anna share $10 with a ratio 2:3. This means that for every $2 Chris receives, Anna will get $3." The cases she considered included "How much money do they each have if the total is $35?" and, finally, "If Chris gets $10 how much will Anna get?"

Whereas both teachers described the multiplicative relationships, Clare's multiplying factors, though simple, appear more arbitrary and thus, perhaps, more representative or exemplifying than the special cases of multiplying by 10 and doubling used by Hilary. Their very straightforwardness and Clare's explicit tabulation of the operations allowed students to attend to the process rather than the computation. Hilary's class may have benefited from an extra example in which the multiplication and/or division was both more generic and made more explicit. An example like "If 18 boys are in the playground and the ratio of boys to girls is 3:4, how many girls are in the playground?" might allow students to focus on the multiplicative relationship. Relating the number of boys (3) in the ratio 3:4 to the number of boys (18) in the playground requires specific identification of the need for a multiplying factor (6), and that it is the operation of multiplication that is required. The too-special cases of doubling and "add a zero on the end" may obscure such multiplicative relationships.

Working on worded problems

Towards the end of both pairs of lessons the teachers gave students similar worksheets comprising problems chosen but not created by the teachers. Some problems required careful application of proportional reasoning, to parts and to the whole, in real-world contexts. In Hilary's case the first was relatively straightforward: "If the ratio of cups of water to cups of oatmeal is 3:1 then how many cups of oatmeal are required if 12 cups of water are used?" Most students managed this alone, despite not having seen examples in which a specific equivalent ratio needed to be obtained by identifying the appropriate multiplying factor. The next example was harder, stating that the ratio of boys to girls in a swimming club is 7:4, and asking how many boys belong if there are 121 children in the club. There are two sources of difficulty here: first, students need to attend to the total and find the scaling factor from that, and, second, knowledge of multiplication number facts for 11 is required, something not usually addressed in Australian schools. Few students attempted any of the examples after this, as the lesson ended. There was no time for follow-up; Hilary concluded the lesson by saying that they had enough knowledge to do the problems and that they would consider them in a following lesson (not observed).

As Clare's students attempted their worksheet problems she emphasised the importance of using the tabular method she had explained. As they discussed the second example – a concert attended by 400 people in which the ratio of children to adults was 7:3 – a student highlighted that the answer could be checked by adding the obtained number of children and adults to see if it was 400. Another student explained that instead of going directly from the ratio total of 10 to the people total of 400 by multiplying by 40, he had scaled to a total of 100 and then added this four times. Clare recognised that this approach was correct. With the next example, "one in every thirty raffle tickets wins a prize", Clare highlighted the importance of attending to the wording to identify the categories defining the parts (winning tickets and losing tickets) and the total (30).

Here the examples used on the two worksheets were similar in complexity; what was different was the preparation that students received before tackling them. Hilary had been quite teacher-directed in her earlier teaching but she did not demonstrate how to tackle application problems in advance of assigning them to students. Moreover, the lack of explicit discussion about multiplicative relationships may also have hindered students' efforts. Clare, in contrast, had established the tabular method in earlier examples, and emphasised it again as students worked. Furthermore, she clearly identified the parts, the total, and the operations required, and could also judge the correctness of students' alternative methods.

Mathematical knowledge evident

It is illuminating to consider the similarities and differences between the two lessons. In many respects the examples used – both in terms of their structure and their numerical complexity – were strikingly alike.

Both classes had activities that compared discrete collections and requested the simplified ratios, and the problems on the worksheets appeared of equal difficulty. Both teachers knew their students' mathematical capabilities well enough to select tasks with appropriate cognitive demand. This kind of knowledge seems to fit within two of the constructs suggested by Ball and others (see Hill *et al.,* 2007), namely "knowledge of content and students" and "knowledge of content and teaching", the former because of the requirement to know how students learn content and what constitutes an appropriate task for their current level of understanding, and the latter because of the need to choose appropriate examples that guide students towards correct mathematical ideas.

Clare provided a greater selection of examples in the discrete collections tasks, with a larger range of factors; in Gibson's terms these examples appeared to provide sufficient affordances for inductive generalisation of the operations for simplifying. Hilary's examples involved fewer, more basic factors, possibly limiting their capacity to afford a clear generalisation of the principles. On the other hand, her use of combined sets yielded ratios that could not be simplified, which Clare did not have, thus providing non-examples that may have illuminated the general principle because of what they were *not*. This knowledge is both mathematical and pedagogical, and probably fits best in the category of "knowledge of content and teaching". What may be more salient, however, is what this examination highlights about the range of examples teachers could use, and how variation affords the exemplification of different key ideas. Such a collection of tasks should include examples with a variety of simplifying factors, examples that do not simplify, an example of 1:1, and ratios that are different from each other but equivalent.

Both teachers exhibited adequate "common content knowledge" (the content that needs to be taught to students), although Hilary's representation of 1:5 as a fifth was problematic. Clare's structural method of representing and operating on ratios seemed to provide students with a better strategy for dealing with the typical ratio problems on the worksheet. This knowledge of representations is included in the "knowledge of content and teaching", but the need to formally teach *a particular* method for solving ratios problems – whatever that method might be, provided it is correct and general – seems to fall into the category of "common content knowledge". The absence of such an approach in Hilary's teaching seemed telling when her students encountered the worksheet problems.

The most marked differences between the lessons occurred in the introductory sections. Clare's use of the rectangle colouring scenario, with its continuous context and the way she constructed the example so that her smaller ratio (1:2) naturally produced an equivalent ratio (2:4), made it much easier to make a conceptual link between equivalent ratios and simplification. Clare's third example, in which students found colourings of different-sized rectangles, all of which showed 1:3, likely helped students to further appreciate these ideas. Hilary certainly understood this content, but there was no inherent reason, with the straw examples that were Hilary's main introduction to equivalence, to consider why you might think about 10:15 as 2:3. Here, again, we have the category of "knowledge of content and teaching", but our examples highlight the complexity of this category. Not only does it matter what numerical examples are chosen, but the context in which they are situated (in this case, continuous versus discrete) may be significant.

One notable aspect of Clare's lessons was her capacity to make connections fluently among a range of mathematical topics. She made reference to area understanding (and also mentioned perimeter), made clear links to fractions, and used correct terminology such as "factor". This "knowledge of connections" within and across topics may also be regarded as both "common content knowledge" and "knowledge of content and teaching", but such a classification seems to obscure one of the few factors associated with teacher knowledge that have been found to have an influence on student outcomes (see Askew, Brown, Rhodes, Johnson, & Wiliam, 1997).

Concluding observations

Whereas we learn much about the mathematical knowledge of these teachers by observing these lessons, unfortunately, we only see what was actually displayed. As Hill and her colleagues point out (2007), our judgements are influenced by our own mathematical knowledge as observers, and nor can we generalise about these teachers' broader mathematical knowledge for teaching from so few observations. However, such observations serve two purposes: they reveal what knowledge is evident in the teachers' practice, and they provide a stimulus to the external observer to question what knowledge is desirable and what role alternative examples might play. They are thus informative and educative.

Furthermore, these few lessons have highlighted the complex knowledge that is required to choose and use examples effectively. Indeed, underlying the construction of the examples is one of the Hill *et al.* (2007) constructs that has not been mentioned (although it was evident when Clare appropriately judged a student's correct alternative approach), namely "specialised content knowledge". Constructing an example that illustrates the desired principle is not something that is taught directly to students learning about ratio and so is not "common content knowledge" (although some students may be able to do it), but it is essential in the art of teaching. Finding the right context, choosing appropriate numerical values, ensuring that the relevant is obvious and that the irrelevant can also be identified as such, and making the whole situation accessible for students, requires a deep mathematical knowledge. In this case, perhaps the most significant difference between the teachers was that one of them was able to construct examples in which the idea of equivalent ratios arose naturally. (The other significant differences were that Clare had a well-structured representation for dealing with ratio problems, and was more explicit about the part-part-whole comparisons.)

There is an underlying assumption here that examples facilitate learning. If so, these issues associated with constructing examples, identifying their affordances, and using them to best effect in the classroom perhaps need to be addressed more explicitly with teachers, in pre-service training and professional development. It may be that examples of examples may help teachers identify some of the critical aspects, and further develop their own mathematical knowledge in the process. (The meta-question about how to construct examples that will help teachers understand the mathematical knowledge for constructing examples will be left for another paper!) If this can be done, then perhaps the examples that teachers use will become better exemplifiers of general principles.

Acknowledgements
This research was supported by Australian Research Council grant DP0344229. My thanks go to Monica Baker and Kiri Harris who were research assistants on this project, and especially to the teachers in the project who allowed us into their world.

Note
[1] A superset of the data from this paper was reported by Chick and Harris (2007).

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Analyze and identify what mathematical steps produced each of these incorrect solutions.

(a)	(b)	(c)
49	49	49
x 25	x 25	x 25
405	225	1250
108	100	25
1485	325	1275

(submitted by Deborah Ball)