EXCESS OF GRAPHICAL THINKING: MOVEMENT, MATHEMATICS AND FLOW

WOLFF-MICHAEL ROTH

In mathematics education, the correspondence between external expressions (such as speech, written mathematical forms, and gestures) and some internal mechanism generating these expressions is presupposed theoretically and methodologically (*e.g.,* in interviews, tests, examinations). Even in enactivist accounts, the emergence of mathematics is explained in terms of the coupling of organismic and environmental structures. In contrast, epistemological approaches that give primacy to movement (*e.g.,* Sheets-Johnstone, 2009) do not presuppose pre-formed structures as underlying the various movements involved (*e.g.,* in graphing specifically and mathematical thinking generally). The movement of thinking is theorized in purely immanent terms that require no transcendental aspects (*e.g.,* concepts, schema, mental framework, representation, and the likes) to unfold. At best, some conscious thought is a trigger that sets thinking off to unfold on its own (Luria, 2003). To stimulate mathematics educators to theorize mathematical thinking in new ways, I provide three case studies, in which mature scientists, employing graphs, *find* their thoughts in what they have mathematically expressed, which, in all these cases, turns out to be such that something has to be erased and redone differently. My purpose in this article is to offer the beginning of an alternative epistemology. The essence of the proposed approach is this: mathematical thinking is a form of movement that grasps itself only in its results rather than being prefigured in schemas, constructions, representations, or conceptions.

Some received ways of considering thinking related to graphs

The movement-based approach to graph-related thinking differs from the ways in which graphing tends to be presented. A typical approach, especially used in the analysis of gestures in mathematics (education) contexts, employs image schemas and enaction, for example, that of natural continuity, which is based on the source-path-goal schema that is said to allow lines and curves to be conceptualized in terms of the motion of an object tracing the trajectory (Núñez & Lakoff, 1998). This approach permits professional mathematicians to use expressions such as "growing, moving, oscillating, approaching values, and reaching limits" (p. 89). In a typical case, where he studies the presentation of Guershon Harel in the process of showing his mathematics education colleagues a proof, Núñez (2009) describes how Harel draws the line for the function $y = x$ and then explicates: "As he says 'increasing,' he gestures with his right hand (palm down), with a wavy upward and diagonal movement (slightly along the line $y = x$). This gesture is co-produced with the enactment of a source-path-goal schematic notion" (p. 320). Here, the schematic notion is said to be *enacted*, the source of whatever mathematical cognition in the mind comes to be displayed and evidenced in a social arena.

In another classical text on the cognition relative to graphs and graphing, the authors provide a model for the mathematical cognition of Herbert A. Simon, Nobel Prize winner in economics, in the process of explaining a supply–demand graph to a student (Tabachneck-Schijf, Leonardo & Simon, 1997). As part of his explanation, Simon constructed a graph, pointed to parts of it, and provided a verbal explanation. The authors then describe what Simon would do if he wanted to explain why a price above the equilibrium point would be driven back to it: "He does this by considering (in a pictorial representation) a higher price, *showing* that at this price there would be a surplus [...] and then reasons *verbally* that the price would decline" (p. 321). The authors conclude that in this explanation, integrating the verbal and gestural modalities, Simon "used both pictorial and verbal representations in his explanation" (p. 321). Elsewhere in the study, the authors show how imagistic representations are stored in "pictorial long-term memory" and, when required for drawing the graph, are pulled into "pictorial short-term memory"; on the basis of the short-term memory contents, the graphing expert Simon draws the graph. The verbal long-term memory is connected with the perceptual long-term memory so that the relevant information is pulled into short-term memory leading to the production of the verbal part of the explanation. This study is consistent with other investigations, for example, on the production of iconic gestures (the kind that Núñez discusses) as outward equivalents of inner images.

One approach that goes beyond others in describing learning as the emergence of mathematical knowing is grounded in the enactivist literature (Proulx, 2013). This author describes the organism (learner) and environment in a process of co-evolution because of their interaction, but he already takes the organism and environment as identifiable parts that *inter*act. For Proulx, thinking is the result of a structural coupling: "The students' explorations or strategies are oriented by their own understandings and meanings of these situations and tasks and by what constitutes issues for them to explore through these 'triggers'" (p. 314). In the example of how students in a mathematics teacher education course mentally solve $x^2 - 4 = 5$, Proulx notes that one strategy existed in "finding the values of x that give a null y-value, or what is commonly called finding the zeroes of the function where the function intersects the x-axis at $y = 0$"

For the Learning of Mathematics **35**, 1 (March, 2015)
FLM Publishing Association, Fredericton, New Brunswick, Canada

(pp. 319–320). He describes the emergence of the strategy for the student after transferring the 5 to the left. He describes strategies as "emergent, brought forth at the moment of meeting with the task, at the moment of engagement with it, at the moment of posing it" (p. 316). That is, what emerges is consistent with the environment, the prior understanding of the student, the "meanings" s/he imbues. But in this, Proulx's description is incompatible with other analyses, which suggest that in emergence, the structure following a bifurcation cannot be predicted based on prior structure. What an emergent account of mathematical behavior requires is a description of—and categories for doing so—the transition between two radically different structures, where the point linking the two belongs to two very different orders (Roth & Maheux, 2014). That is, in all the examples Proulx provides, *emergence* of new thinking is not described, nor is the independence of the thinking strategy from the prior state (understanding), which would have left the student in a situation of surprise about the results of his actions. Movement theorists, on the other hand, are in the position to theorize this aspect because animation "is no longer regarded the mere output but the proper point of departure for the study of life" (Sheets-Johnstone, 2009, p. 214). Surprise, therefore, marks a dissensus of sense (as in sensation) with itself (as meaning) (de Freitas & Sinclair, 2014).

All these approaches are based on models in which something on the inside, a conception, construct, mental representation, idea, thought, schema, or structure *drives* or (co-)*determines* the actions of the knower or is, in the knowledgeable production, externalized for the purposes at hand (*e.g.,* an examination, interview, lecture). Although there are scholars who orient mathematics educators towards the mobility of mathematics (*e.g.,* de Freitas & Sinclair, 2013), they tend to do so through the finished products and, thereby, precisely miss what is at stake in the approach presented here. As one study shows, the trace of movement remaining in diagrams only stands for a tiny fraction of the thinking in movement exhibited in gestures and body movements (Roth, 2012). Some philosophers, social psychologists, and neuropsychologists have made very different statements concerning thinking: the knower *finds* his/her thought in what s/he has produced. Thus, for example, thinking "is not a representation associated or linked externally with the movement itself, but is *immanent* in the movement inspiring and sustaining at every moment" (Merleau-Ponty, 1945, p. 127, emphasis added). Moreover, "thinking [*mysl'*] is not expressed but takes place/is completed in [*soveršaetsja*] the word. We can therefore speak of the becoming […] of thought in the word" (Vygotskij, 2005, p. 963, my translation). Vygotskij therefore writes about the dual movement [*dviše-nie*] between, and giving rise to, thinking and speaking in movement.

Finding thought when the movement of thinking has stopped

In this section, I present three case examples of scientists in the process of talking/ lecturing with graphs. In each case, we observe these experts first produce aspects of graphs and then erase them again, producing something else in their place. If image schemas or internal representations had driven these productions, the experts should have produced the correct graph in the first place. On the other hand, if these experts find their graph-related thoughts in what they have produced, then we require a different account of mathematical thinking than by means of structures that produce thinking.

Three case studies of graphical thinking

The first case derives from a scientific research meeting during which the team members analyze and talk about the data collected so far. Their data collection involves measuring the amount of light that is absorbed in the photoreceptors of some fish species, where the absorption is due to interaction with vitamin-A_1-based rhodopsin and vitamin-A_2-based porphyropsin.

At one point, the team leader, with over 30 years of experience in the area, asks about the quality of the data and then suggests that there are some *preconceived* notions about how the half-maximum bandwidth of a near-Gaussian absorption curve varies with the amount of porphyropsin in fish eyes. He then gets up, walks to the board, produces the interjection "um" and stares at the board for 3 seconds (Figure 1a). He writes "1/3" while saying "a half" (Figure 1b), stops, then emphatically writes a 2 over the 3 followed by an "m" without saying anything (Figure 1c). He then erases what he has just produced (Figure 1d) and writes "HMB" while saying "half max," pauses (Figure 1e), before finishing "bandwidth." He draws two lines orthogonal to each other, which *will have been* the ordinate and abscissa of a graph, while saying "versus." He then writes A_1/A just below the intersection of the two lines (Figure 1f), stops, and pauses; he then erases what he has written (Figure 1g) and writes A_2/A_1 in its place (Figure 1h). He steps back, gazes in the direction of what he has written, then steps forward again. He erases what he has written (Figure 1i) and writes "$\%A_2$" approximately below the center of the horizontal line (Figure 1j). Over the course of 4 seconds, he gazes left, right, left, right, left, right, and then left again before writing 100 on the left end of the horizontal line while saying "from say a hundred" and then writes "0" accompanied by saying "to zero" on the right end. With further pausing and shifting gazes, he finally draws a curved line while saying "you'd expect something like this" (Figure 1k). The doctoral student present in the room suggests that the curve really looks different, while

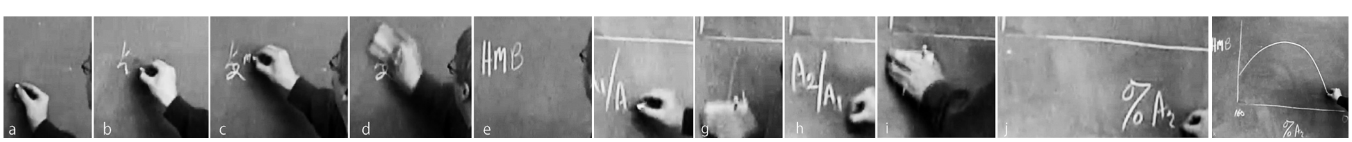

Figure 1. A professor with research scientists writes and erases multiple times before the "preconceived notion" actually appears on the chalkboard.

the scientist-professor returns to his seat and copies what he has just noted on the board—as if he needed to copy what he has just produced.

The second example derives from a third-year university physics course in optics. In the context of a student question about blackbody radiation, the instructor, with over 20 years of experience teaching physics, talks about the relation between temperature and color in the context of the production of steel. He talks about the problem of hardening steel by bringing it to a high temperature: it is very brittle. This requires the machinist to temper the steel. He steps to the board, sketches Cartesian coordinates, and says "What you have done in terms of temperature is." He then draws a line while saying "you bring the steel up to the temperature here," then stops, marks off the abscissa, writes an "R," moves his hand to the end of the previously drawn line (*i.e.,* the upper right end of the "diagonal"), stops, moves back down to the abscissa, marks off another point, and writes "Y," while saying "in the red to yellow range." He moves his hand back to the upper end of the line and, apparently connects the beginning of what comes to be a vertical line, while saying "and then you quench it, you cool it off really quickly." He continues by saying, "You re-heat the steel, you bring it up to what is called the straw color," and marks off a third point to the left of the "R." He then begins drawing another curve, which initially moves parallel to the original one (Figure 2a) but then turns downward (Figure 2b). The instructor stops and wipes off part of the curve in moving his hand over the chalk line backward (Figures 2c and 2d). He then draws a new turn (Figure 2e) and returns the curve to the abscissa to the left of what he has marked as "the straw color" (Figure 2f). (Because of the physical laws of blackbody radiation that relate color and temperature, the return curves should be the same as the heating curves, and the relationship between temperature and color is not linear.)

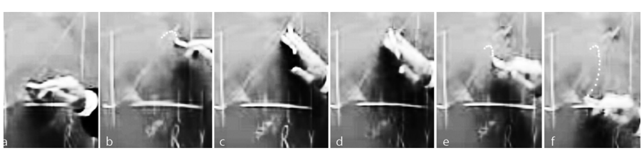

Figure 2. A graph is drawn, erased, and redrawn as a lecturer explains the hardening of steel.

The third example derives from a second-year university course on thermodynamics. After having explained how an air conditioner works, the veteran professor (45 years after his PhD) moves to explain the working of a refrigerator in terms of a Carnot cycle graph, repeatedly used in this course, before relating pressure, volume, and temperature. After having produced the graph—including signs for the inside (food) and environment and the energies Q_1 and Q_2 transferred from the food into the environment—the professor moves to writing two equations "$Q_{in} = Q_1$" and "$Q_{out} = Q_2$." (Figure 3a). He then writes "$Q_{in} > Q$" while saying "Q in is greater than Q out" (Figure 3b). He stops and gazes at his writing, producing the interjection "um." Three seconds later, he erases the "in" from the left "Q" (Figure 3c) and writes "out" in its place (Figure 3d). He then writes "in" next to the Q on the right side of the inequality. He steps back about 2 meters and, like a painter with respect to her latest brush stroke, gazes at the equation as if taking in what he has just produced, before moving on to talk about the internal energy to the system. Only seconds after, he writes below the inequality "frid" while saying "oh that's fine, fridge is" (Figure 3e stops, erases what he has written (Figure 3f), and then writes "refrigerator" in its place while saying "no that is the refrigerator" (Figure 3g).

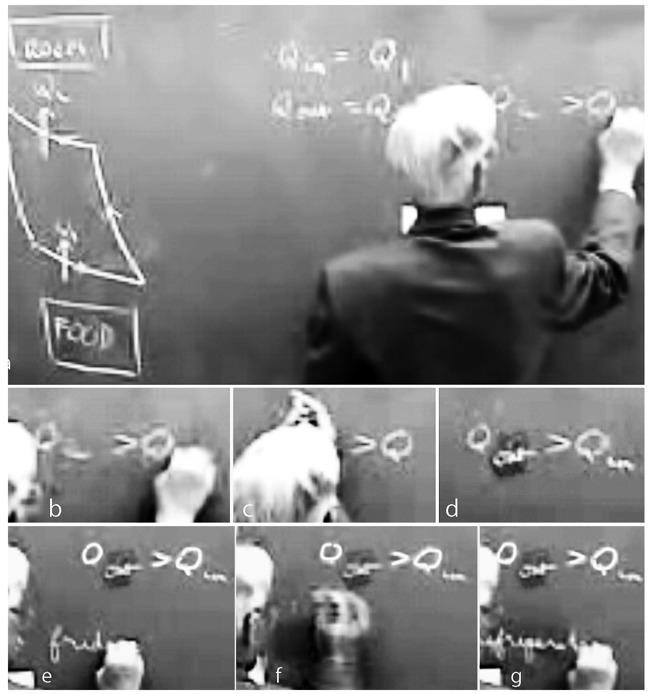

Figure 3. Equations are written and erased in a thermodynamics lecture.

In all of these instances, highly educated and experienced scientists/professors are in the process of talking about topics that they are familiar with from their research or teaching, employing mathematical forms (graphs, equations). Taking any current theoretical frameworks about mathematical cognition, we would expect them to have formed representations, schema, conceptual structures, and the likes through their extensive scientific experience. During such presentations, they are said to externalize these constructions or enact underlying schema. Why then, after so many years in the field, are they seen to produce parts of graphs and the mathematical relations going with them *only to erase* what they have done and to replace what they have written? Sometimes, this repeatedly takes place before a final version remains on the chalkboard. Traditional cognitive accounts tend to introduce interference processes that degenerate in some way the output of knowledge between where it is encoded and when it arrives on the public scene (*e.g.,* slips of tongue, misfirings, defense mechanisms) (*e.g.,* Rodd, 2013) [1]. Although more refined accounts do emphasize the emergent nature of mathematical productions that results from structural coupling (Proulx, 2013), deficit-oriented accounts have not explained (to my knowledge) such events as *positive* phenomena in the way observed here; such positive phenomenal descriptions are provided by kinetic accounts, as recent studies of forgetting in aircraft cockpit have shown (*e.g.,* Roth, Mavin & Munro, in press).

Reading the cases: cognition in movement is the movement of cognition

The approach I advocate here is one in which movement "knows" nothing but itself. Luria (2003) suggests that we need to understand persons expressing themselves in terms of *kinetic melodies,* movements, which unfold on their own. Kinetic melodies, movements, reproduce themselves such that no conscious effort is required or necessary: it is a phenomenon of pure immanence and invisible as such (Henry, 2000). The result of such a consideration is that there is an independence of the movement from any conscious awareness thereof, because awareness requires explicit, transcendent forms that allow the individual to make the movement *present again* to represent it. The movement is immanent to itself (Henry, 2000). In a movement approach, conscious awareness, plans, thoughts, or intentions *do not determine* the movement but *follow* them. The relation between a plan or intention and an action can be described only with hindsight. Some trigger sets mathematical thinking into movement in Figure 1, which results in something like 1/3 that leads to a halt, followed by a new movement, the result of which is a bold-faced 2 over the 3 with the result of ½. Thinking can grasp itself only through its results, having become object. Thus, the thinking movement continues producing an "m," leading to a new halt, pause, and then into a movement the result of which is an erasure. Sometimes, the result of the new movement is (conceptually) not or little different, such as in the movements (in Figure 3) that first led to "frid," erasure, and then to "refrigerator."

It has been asserted that even in adults, the relation between thought and word is not constant but consists in a "movement from thought to word and reversely, from word to thought" (Vygotskij, 2005, p. 962, my translation). This movement—*i.e.,* the relation between thought and word—constitutes an evolutionary "development in the true sense of the word" (p. 963). Vygotskij also suggests that thought is completed in the word, which I extend here to any sign a speaker might produce. Because it is completed in the sign (though not exhaustively), thought thereby might find something in the sign that did not exist prior to its completion. In this case, thinking finds something of itself that exceeds what it has thought prior to the expression (sign). This then makes intelligible why even the most competent individuals may find themselves erasing the graphs and mathematical symbols they have just drawn or written. Thus, in the case where the biologist restarts labeling the axis twice (Figure 1), which produces, in sequence, the axis labels A_1/A, A_2/A_1, and $\%A_2$ (in a different location), it appears as if the intention is found in what has been produced: *the biologist knew what he was looking for when he saw it.* Thus, after the first inscription, the biologist erases it only to launch into another movement, the result of which is A_2/A_1. But, as the events show, this is not what he apparently intended, or what the "preconceived notion" consists in. Again, he erases what he has just inscribed on the board to launch into yet another movement, which now places a different sign in a different place with respect to the existing contents of the chalkboard.

In these examples where something produced is more-or-less immediately erased, it is not sufficient to say that some inscription is the result of a dialectic between subject and task setting (as Lave *et al.,* 1984, have done) or the result of a structural coupling between the agent and environment (as Proulx, 2013, has done). Any supposed prior structures that are part of the production would also have to be part of the recognition of its inappropriateness.

A movement-based approach recognizes the generativity of moving thinking that is *in excess of itself,* so that chalkboard lines may emerge following and as the result of a series of hand/arm movements in which the thinking finds what it wants to do. Unintended thinking movement is *pregnant* with the new, invisible, and unforeseen (Merleau-Ponty, 1964). An example of this is provided in Figure 4, from the thermodynamics course. In the process of showing how a substance can be cooled by means of adiabatic magnetization and demagnetization, the professor has started a graph. On it appear two lines showing the beginning of a process: the first corresponds to a change in the magnetic field from $B = 0$ to $B \neq 0$ at constant temperature T but changing entropy S; and the second from the latter to the former magnetic field at constant entropy S but lowering the temperature T. After drawing the second line, the professor has stepped back, gazing at the diagram on the chalkboard (Figure 4a). The series of images then shows

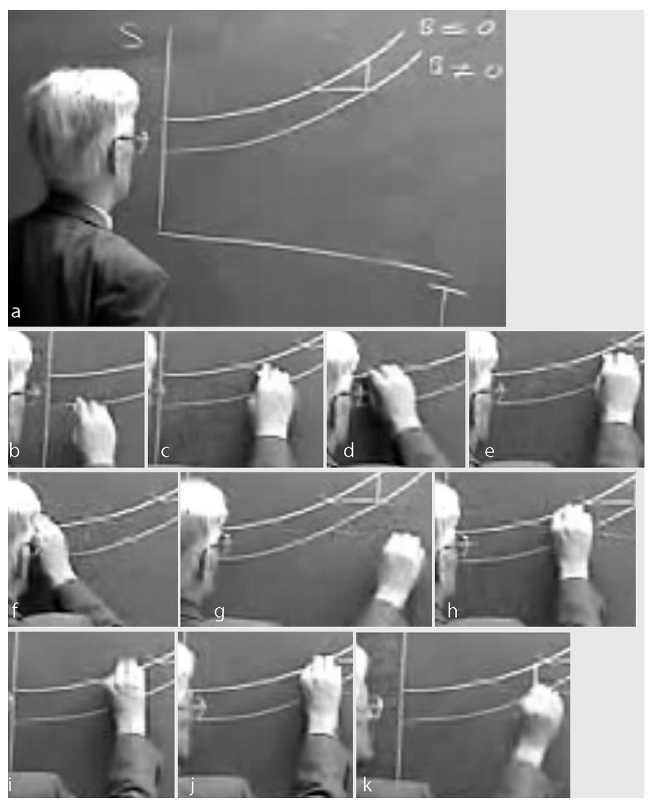

Figure 4. *Multiple hand movements precede the actual drawing of the line, as if these were structuring the perceptual space.*

how he steps forward with the chalk towards the left part of the graph (Figure 4b–e). This is followed by a movement to the right, back to the left, and to the right again. Another movement to the left ensues (Figure 4f), at which point the chalk touches the board so that the rightward movement produces a very fine line (Figure 4g). From there, the hand moves to the curve $B = 0$ (Figure 4h), stops, twists a few times left and right (Figure 4i), then moves to a different spot on the $B = 0$ line (Figure 4j) before drawing a vertical line corresponding to the first he had drawn (Figure 4k). It has taken 14.8 seconds, from the first to the last image in Figure 4. It is as if all the movement and gazing is in fact preparing the space within which the line that would finally have been drawn could emerge. Even the hand movements preceding the drawing of the thin (fine) horizontal line do not appear to be symbolic gestures, which would stand in for something else. At best, these movements are random, searching movements that will allow the agent to find something like his thought.

This event also highlights what is already apparent in the three preceding examples, namely the temporality involved in movement and, therefore, in anything of what we know as or denote by mathematical cognition. We also see again how the person, after having moved, steps back, gazing at what he has done, like the painter stepping back to see what the preceding stroke gives. This is consistent with a (creative) person who does not know beforehand what his/her thinking in movement gives until after it is completed and has been gazed at (pondered about?).

In the foregoing examples, we see the beginnings of the movement out of an initial stance, which, according to some, is the source of the emotion from which springs animation (movement) (Leont'ev, 1959). The biologist stands before and gazes at the empty chalkboard, as if gathering himself and then allowing the movement to unfold. Stepping back is another form of taking a stance with respect to the drawing, leading to a reflective attitude and movement. It is also here where a gathering takes place, a launch pad for the thinking movement that arises from it, such as seen in Figure 4 when the professor returns to the chalkboard, which will have resulted in the drawing of a vertical line. The stepping back and forward again, pausing that prepares the launching into the second phase where the drawing and speaking movement is related to the reheating of steel, exhibits the pulsing of the movement. It is not homogeneous, as seen in Proulx's (2013) account of the emergence of mathematics, where the movement appears to emerge but without specific temporality proper to its dynamic. We also see changing intensities, which would be central to the dynamic approach concerned with movement and animation. Thus, for example, the initially more faintly written "3" comes to be overwritten by a much stronger, larger, and more bold 2. The "HMB" that appears in the place of "½" is emphasized even more (Figure 1). Similarly, there are differences in strengths of the lines that the optics lecturer draws, the stronger lines marking out the grid and letters, a less strong line produced while talking about the hardening of the steel, and an even more faint line standing for the strengthening of the steel. Where in the enactivist and embodiment accounts, concerned as these are with concepts, do we find the role of intensities?

This account is most similar to an account of graphing specifically and of mathematics generally (*e.g.,* Proulx, 2013). It differs, however, in the accentuation scholars like Proulx give, who emphasize *structures* and *structural coupling*. In the enactivist approach, mathematics springs forth from the interaction of organismic and environmental structures. In movement-based approaches, the production of the subject and its object (environment) is itself part of this unfolding transformation of the world. The subject and its environment are the outcomes rather than the inputs in to the structural *coupling* that Proulx describes (see below). Those who give primacy to thinking in movement emphasize the unique totality in which immanent movement is but an irreducible moment. This shifts the emphasis, but not because of structural coupling. The relation is well explained in an analogy with musical performance, in the case of which it is possible to speak of a connection between two structures: instrument and player (Ingold, 2011). Although instrument and player bear on each other, "the line of the melody *does not lie in the connection*" (p. 83, emphasis added). It runs transversally to the connection, like the river runs transversally to the bridge that connects the two riverbanks. It is not in the coupling of the two structures, not in the connection between them that the line of the melody lies. The melody (movement) runs transversally to the structures and their connections. Thus, the line of movement "continually issues forth from that place, in the midst of things, where the fiddler and the violin are conjoined in a passionate embrace" (p. 83). A dynamic approach needs to show how the inner and outer, organism and environment come to be separate in the first place. This occurs when the inner (organism) and outer (environment) are not merely coupled but where the very distinction between inner and outer is the result of movements.

From thinking determined by structures to thinking in movement that reproduces itself

None of the received approaches accounts for the fact that somewhere in human evolution or individual development, thinking emerges before any such structures exist. Movement theorists, such as Sheets-Johnstone (2009) show where other accounts fall short: (a) in their failure to link affect and animation; and (b) in the failure of doing justice to the spatiotemporal dynamics of affect and animation. The prefix "en-"—as the "em-" in *embodiment*—derives from Latin *in-*, used to form verbs that denote putting something into whatever noun follows the prefix. Thus, to embody describes putting something into a body; to enact is used to put something into action. But, as animate theorists point out, movements do not enact anything but produce and reproduce themselves: they are completely immanent and do not require transcendent structures to make them do what they do. The starting point of my consideration lies in the recognition of the advances scholars such as Proulx (2013) or de Freitas and Sinclair (2013) provide over other theories of mathematical knowing and learning. But we need to take some steps that take us further to give primacy to thinking as movement. The three cases discussed in this article allow us to work towards developing a theory of mathematical cognition that takes into account the fact that the mathematical problem solver apparently does not really know what he his doing until after he has done it.

In fact, the professor in the thermodynamics course, after he had completed the graph depicting the changes during the magneto-caloric effect stops, gazes at the graph, and then while shaking his head and walking to look at his notes says, “I think there is something wrong with this picture.” In that instant, he does not recover what it is that is wrong. It is only later, while gazing at some equations that he had produced and talking about the implications thereof, that he notes, “And the second consequence is,” then pauses, turns to his left, gazes at the graph, and says, “this is wrong.” He then proceeds to produce a graph that is indeed consistent with thermodynamics. Again, the new unpredictably bursts forth in excess of the structures previously articulated—or the professor would have produced the correct graph before.

There already are laudable attempts to bring agency into mathematics education. But I tend to agree with others that agency does not bring things to life, as if a sprinkling of agency could “restore to life the generative fluxes of the world of materials in which they came into being and continue to subsist” (Ingold, 2011, p. 29). I suppose that mathematics educators would want to arrive at an approach that does not just explain the emergence of some mathematical behavior but that explains how whatever arises is new to or unintended by the agent. Intent is like the connection (coupling) of the two opposite banks of a river by means of a bridge. But the really important thinking in movement, that which expresses the becoming (developing) of a person’s life, is not well expressed in this approach, just as the flow of the river is not described by the bridgeheads and the bridge. The flow is not laminar between the riverbanks. Instead, there are back-eddies (*contre courrants*), retrograde movement, back-currents, crosscurrents, and rip currents. The river is moving slow, in parts, and fast in other parts; there are places of non-moving water. And the flows are different viewed transversally across the flow. Intransitive relations are orthogonal to the bridgeheads on opposite banks of the river; these are relations within the river flow itself. Ingold (2011) uses the analogy with genome in biosciences, according to which the (internal) image does the same kind of work in perception as the genes do to the (outward) phenotypic expression. He suggests that to “regain the currents of life, and of sensory awareness, we need to join in the movements that give rise to things rather than casting our attention back upon their objective and objectified forms” (p. 97). Different points of departure are available in the concepts of flow, the *lignes de fuite* (lines of flight, flowing, leaking: Deleuze & Guattari, 1980), and animation (Sheets-Johnstone, 2009). In all of these alternate approaches, mathematical thinking in movement remains immanent to itself; and there are unbridgeable gaps between it and how it appears in transcendent forms (mathematical schemas, representations, constructions, or conceptions).

Notes

[1] Deficit explanations do not explain perceptual phenomena, as Merleau-Ponty (1945) shows in many examples, arriving at descriptions of cognition consistent with modern neurosciences. The phenomenology of perception does provide *positive* accounts by focusing, for example, on the differences in the relation between subject and un/seen object (*e.g.,* Roth, in press).

References

de Freitas, E. & Sinclair, N. (2013) New material ontologies in mathematics education: the body in/of mathematics. *Educational Studies in Mathematics* **83**(3), 453–470.

de Freitas, E. & Sinclair, N. (2014) *Mathematics and the Body: Material Entanglements in the Classroom.* Cambridge, UK: Cambridge University Press.

Deleuze, G. & Guattari, F. (1980) *Mille plateaux: capitalisme et schizophrénie.* Paris, France: Les Éditions de Minuit.

Henry, M. (2000) *Incarnation: une philosophie de la chair.* Paris, France: Éditions du Seuil.

Ingold, T. (2011) *Being Alive: Essays on Movement, Knowledge and Description.* London, UK: Routledge.

Lave, J., Murtaugh, M. & de la Rocha, O. (1984) The dialectic of arithmetic in grocery shopping. In Rogoff, B. & Lave, J. (Eds.) *Everyday Cognition: Its Development in Social Context,* pp. 67–94. Cambridge, MA: Harvard University Press.

Leont’ev, A. N. (1959) *Problemj razvitija psixiki.* Moscow, USSR: Akademii Pedagogičeskix Nauk.

Luria, A. R. (2003) *Osnoby nejrolpsixologii* [Foundations of neuropsychology] Moscow, Russia: Isdatel’skij Centr Akademija.

Merleau-Ponty, M. (1945) *Phénoménologie de la perception.* Paris, France: Gallimard.

Merleau-Ponty, M. (1964) *Le visible et l’invisible.* Paris, France: Gallimard.

Núñez, R. E. (2009) Gesture, inscriptions, and abstraction: the embodied nature of mathematics or why mathematics education shouldn’t leave the math untouched. In Roth, W.-M. (Ed.) *Mathematical Representation at the Interface of Body and Culture,* pp. 309–328. Charlotte, NC: Information Age Publishing.

Núñez, R. E. & Lakoff, G. (1998) What did Weierstrass really define? The cognitive structure of natural and – continuity. *Mathematical Cognition* **4**(2), 85–101.

Proulx, J. (2013) Mental mathematics, emergence of strategies, and the enactivist theory of cognition. *Educational Studies in Mathematics* **84**(3), 309–328.

Rodd, M. (2013) Teacher visualization loss mid explanation: an issue when teaching geometry. *Proceedings of the British Society for Research into Learning Mathematics* **33**(3), 49–54.

Roth, W.-M. (2012) Tracking the origins of signs in mathematical activity: a material phenomenological approach. In Bockarova, M., Danesi, M. & Núñez, R. (Eds.) *Cognitive Science and Interdisciplinary Approaches to Mathematical Cognition,* pp. 182–215. Munich, Germany: LINCOM EUROPA.

Roth, W.-M. (in press) Becoming aware: towards a post-constructivist theory of learning. To appear in *Learning: Research and Practice.*

Roth, W.-M. & Maheux, J.-F. (2014) The emerging and emergent present: a view on the indeterminate nature of mathematics. *Mathematics Education Research Journal* **26**(2), 325–352.

Roth, W.-M., Mavin, T. & Munro, I. (in press) How a cockpit forgets speeds (and speed-related events): toward a kinetic description of joint cognitive systems. To appear in *Cognition, Technology and Work.*

Sheets-Johnstone, M. (2009) *The Corporeal Turn: An Interdisciplinary Reader.* Exeter, UK: Imprint Academic.

Tabachneck-Schijf, H. J. M., Leonardo, A. M. & Simon, H. A. (1997) CaMeRa: A computational model for multiple representations. *Cognitive Science* **21**(3), 305–350.

Vygotskij, L. S. (2005) *Psyxhologija razvitija čeloveka* [Psychology of human development]. Moscow, Russia: Eksmo.