CONTINUOUS PROBLEM OF FUNCTION CONTINUITY

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Consider the vignette below that gives the essence of a conversation that took place between one of us (G) and a first year university student (S) taking a calculus course.

G: Consider the graph of $g(x)$. Is this a continuous function? (*See Figure 1.*)

S: Yes it is.

G: Why do you say so?

S: Because you can draw it without lifting the pen.

G: Do you think it is continuous at, say, -10?

S: What do you mean?

G: Is it continuous at this point? (*Pointing to a point on the negative x-axis.*)

S: That question doesn't make sense. The function is not defined there. How can I talk about continuity when the function is not there?

G: OK. Let's look at a different graph. Is $h(x)$ a continuous function? (*See Figure 2.*)

S: No, it has a discontinuity at 3.

G: And why do you say this?

S: Because the function is not defined at 3.

G: But, didn't you say in the earlier example that you can't talk about continuity when the function is not defined?

S: Hmmm, but in that example, the function was not there in that whole region.

G: So is it different if the function is not there just at one point?

S: Hmm, I'm confused!

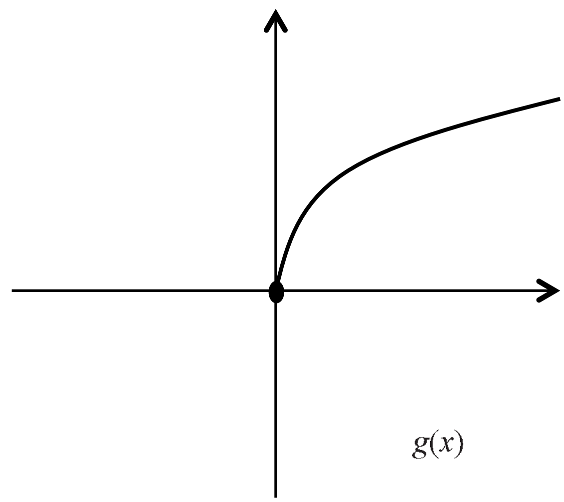

Figure 1. Graph of function $g(x)$.

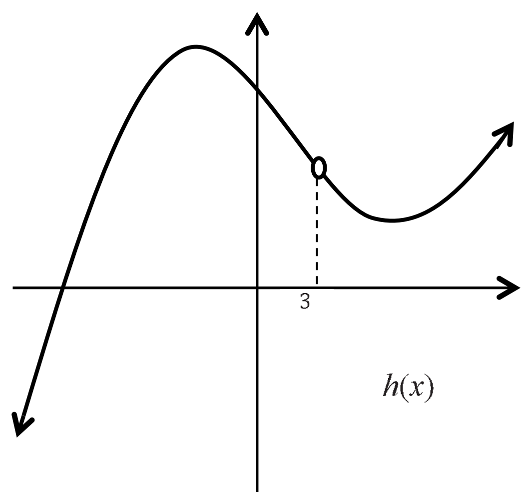

Figure 2. Graph of function $h(x)$.

During interviews with university students enrolled in the same calculus course, it was found that similar confusions about continuity persist in cases where the domain of a function does not include all real numbers. This confusion can be matched with different definitions of continuity that we found in textbooks and various other mathematical sources. We aim to point out two problematic situations that arise through certain definitions (or the lack of certain definitions) involving the concept of continuity. We invite readers to reflect on their own experiences, starting with their personal answers to the question of the continuity of $g(x)$ and $h(x)$, above, and the reasons for these answers.

On mathematical definitions

Definitions are arbitrary, agreed upon conventions (Levenson, 2012). The pivotal role played by definitions in

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mathematics has been emphasized by many writers (*e.g.,* Morgan, 2005; Parameswaran, 2010) and mathematics education literature is abundant with discussions of various aspects of definitions. What seems to emerge out of this literature is the importance of clear definitions, both from a mathematical and a pedagogical point of view.

In the field of mathematics, it is not at all unusual to use different definitions for the same mathematical concept, be it among textbooks, mathematicians or teachers. Often these differences are superficial and nuanced. And more importantly, even if these definitions are superficially different, they are equivalent and consistent. They represent the "same" concept and for the most part imply the same set of properties of that concept. For instance, the definition of a function reads "A rule that assigns to each element in a set A one and only one element in a set B" in the textbook *Applied Calculus* (Tan, 2011, p. 52), while Wikipedia presents an alternative definition as "a set of ordered pairs where each first element only occurs once" [1]. There is also the practice of introducing concepts using simpler versions of definitions at lower levels and then progressively advancing to more rigorous definitions at higher levels. For instance, if we look at the same concept of function, it is often introduced metaphorically at lower levels as a machine that takes an input, x, and returns exactly one output, $f(x)$.

In mathematics education, the need for precise, agreed upon definitions has been discussed in several studies (*e.g.,* Lampert, 1990; Ball & Bass, 2000). For instance, Lampert (1990) showed how misunderstandings arise during classroom discussions when a definition is not precise (this certainly does not imply that when definitions are precise, misunderstandings do *not* arise, but rather, that imprecise definitions may contribute to misunderstandings). While many mathematics education researchers attend to the role of definitions and its importance (*e.g.,* Vinner, 1991; Edwards & Ward, 2008; Robinson, 1962), there are authors who address various issues of definitions in mathematics education, raising pedagogical and mathematical considerations with respect to the definitions of concepts, such as exponentiation, tangent, limit, derivative and symmetry (*e.g.,* Winicki-Landman & Leikin, 2000; Leikin, Berman & Zaslavsky, 1998; Thurston, 1974). There are also research studies on teachers' understanding of definitions (*e.g.,* Levenson, 2012; Leikin & Zazkis, 2010). For example, in their study, Leikin and Zazkis (2010) found that some prospective teachers were unclear of the difference between a definition and a theorem and that knowledge of definitions differed in different content areas of mathematics. Therefore they suggest that more attention should be given to discussion of the notion of definition and its structure and role within mathematics both in school and at university. Vinner (1991) claims that one of the assumptions that textbooks and classrooms make in their presentation and organization of mathematics is that "concepts are mainly acquired by means of their definitions" (p. 65). However, several studies have shown little influence of definitions on students' use of words when they are engaged in mathematical tasks (*e.g.,* Wilson, 1990; Fischbein & Nachlieli, 1998).

In what follows, we discuss two problematic issues related to the definitions of continuity. These issues are intertwined and can be seen as two manifestations of the same problem. The first is the use of the phrase "continuous function" despite the lack of an explicit definition for a continuous function. The second is the mathematically inharmonious and inconsistent ramifications of such definitions when they *are* explicit but incoherent. We then contrast these two issues with other contexts in mathematics, where different definitions for the same concept appear to be in discord with each other.

Continuity: two definitions

The concept of continuity is an important concept in calculus and analysis. It is usually introduced in introductory calculus courses for students who specialize in mathematics as well as for students who do not specialize in mathematics but take calculus as part of their program requirements or their general interest. Continuity is described and defined to suit different audiences at different levels, including the use of intuitive descriptions, informal definitions, formal limit definitions and the more rigorous epsilon-delta definition. In the context of an introductory calculus course, and also in many other common resources, the definitions used for continuity-related concepts are limit definitions. At the outset, a distinction needs to be acknowledged about these definitions. Two different limit definitions (that are labeled as D1 and D2 for reference in this paper) are used for "continuity at a point" (and accordingly "discontinuity at a point") on which the other related concepts of continuity can be based:

Definition D1 [2]:

A function f is said to be continuous at c if,

1. $f(x)$ is defined at $x = c$
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x)$ is equal to $f(c)$

f is discontinuous if any of the above conditions are not satisfied. [3]

Definition D2 [4]:

A function f is said to be continuous at $x = c$ in its domain if,

$$\lim_{x \to c} f(x) = f(c)$$

And f is discontinuous at $x = c$ in its domain if,

$$\lim_{x \to c} f(x) = f(c)$$

The two definitions, in themselves, are inconsistent but this difference between D1 and D2 is subtle and slippery and may not be visible at once. D2 is regarded as more accurate by mathematicians (Tall & Vinner, 1981) while D1 is the more widely used definition in the context described in this article.

The way D1 and D2 are presented in textbooks may not "look" exactly the same as the versions given above. The deciding factor that makes a definition consistent with either

D1 or D2 is the treatment of a point at which the function is not defined. According to D1, a function that is not defined at a point is discontinuous at that point, while according to D2 the question of continuity or discontinuity should not arise. Therefore the difference between D1 and D2 lies more in the way discontinuity (at a point) is defined. It is, however, not our intention in this article to go to mathematical lengths to investigate the accuracy or falsehood of these definitions, but to attend to and elaborate on the discrepancies and consequences of them.

In what follows, D1, the most common definition in introductory calculus, and D2, which is regarded as a more rigorous definition, are attended to in the discussion of the two problems: (a) absence of a definition for a continuous function, and (b) inconsistency of definitions. In the conclusion, we outline how some related concepts of continuity can consistently build on D1 and D2 which may give better visibility to the existing entanglements.

Problem 1A: absence of a definition for a continuous function

We have examined several dozen resources (textbooks, websites, mathematical dictionaries) seeking a definition for a continuous function. In most of the resources, such a definition was not explicitly stated. However, the phrase "continuous function" is loosely used in many places. The absence of definitions can lead to implicit definitions. The following example illustrates how problems can be created due to the absence of a definition for a continuous function.

The topic of continuity starts off, in many textbooks and websites, with the definition of continuity *at a point.* This is the leading definition from which other related extensions to the concept of continuity of a function, each of which has its own definition, may follow (for example, continuity on an interval, a discontinuity/a discontinuity at a point, types of discontinuities, one-sided continuities). Quite remarkably, however, despite the fact that many of these book chapters and websites title the topic of continuity as "continuous functions", they only rarely, proceed to define what a continuous function is. This situation leads to problems. Let us follow the way the situation unfolds in one textbook.

A popular way to introduce continuity in mathematics textbooks (*e.g.,* Neuhauser, 2010; Stewart, 2012) is to start with an example of two functions that agree at all but one point, one of which is continuous (due to the limit of the function at that point being equal to the function value at the point), and the other of which is not continuous (due to the limit being not equal to the function value) at that point. This leads to a definition of continuity consistent with D1. The definition is then operationalized as follows.

To check whether a function is continuous at $x = c$, we need to check the following three conditions:

- $f(x)$ is defined at $x = c$
- $\lim_{x \to c} f(x)$ exists.
- $\lim_{x \to c} f(x)$ is equal to $f(c)$

If any of these three conditions fails, the function is *discontinuous* at $x = c$.

However, the heart of the problem is that these definitions of continuity/discontinuity at a point are not followed by the definition of a *continuous function.* This situation leaves room for students to intentionally or unintentionally construct a meaning for continuous function. Intutively, it is likely that this will be interpreted as "continuous everywhere" which again is problematic. Where is "everywhere"? Everywhere can mean on the real number line or at every real number, which is consistent with D1. In fact, some sources present this interpretation:

> A function that is continuous on $(-\infty, +\infty)$ is said to be continuous everywhere, or simply continuous. (Anton, 1995, p. 105)
>
> A function is a continuous function if it is continuous at every real number. (Mathematics Harvey Mudd College [5])

However, "everywhere" can also be interpreted as every point of the function domain, which is consistent with D2 (*e.g.,* Strang, 1991; Bogley & Robson, 1996). Therefore this situation holds the potential to lead students to construct their own meaning for a continuous function, which could be in discord with the intended definition.

Morgan (2005) quotes the criteria Borasi uses to justify his requirements for mathematical definitions: "A definition of a given mathematical concept should allow us to discriminate between instances and non-instances of the concept with certainty, consistency, and efficiency (by simply checking whether a potential candidate satisfies *all* the properties stated in the definition)" (p. 106). Both D1 and D2 do just this, but the mixing of the two by learners in the absence of an explicit definition for a continuous function can create problems, as exemplified in the dialogue at the beginning of this paper. According to definition D1, $g(x)$, which is the square-root function, is not continuous at -10. In fact, it is discontinuous at all points less than zero as the function is not defined at those points (having "function not defined" as one of the conditions for discontinuity is the very reason why D1 is regarded as not strictly correct) [6]. On the other hand, it appears to be continuous since it has no gaps and can be drawn without lifting the pen, which is a common concept image for a continuous function. As such, the initial claim that the presented function is continuous is consistent with the concept image as well as with D2. Further, the inability to discuss properties of a function that is "not there" is also consistent with D2. Only at the confrontation do the students begin to wonder whether talking about continuity of $h(x)$ makes sense if the function is not defined at a single point.

Problem 1B: absence of a definition for discontinuity

In addition to the absence of the above mentioned definition for a continuous function, some textbooks also do not define a discontinuity (at a point or otherwise) (*e.g.,* Lial *et al.,* 2011; Bauldry, 2009; Begle, 1954). This absence may lead to the confusion that continuity and discontinuity are binary opposites. If $f(x)$ is not continuous at $x = k$, does this necessarily mean that it is discontinuous at $x = k$? Yet again, the outcome depends on the chosen definition. Consider the following analogy: a given number k is not even. Is k then an odd number? While it could be the case, it could also be the case that $k = 3.14$ or $k = 3i$, in which situation the discussion of parity makes no sense, since oddness/evenness is defined only for whole (or integer) numbers.

Problem 2: inconsistent definitions

An additional problem arises when the definition for a continuous function *is* given but is inconsistent with the leading definition of continuity at a point. This is how Wikipedia goes about it:

> The function f is *continuous at some point c* of its domain if the limit of $f(x)$ as x approaches c through the domain of f exists and is equal to $f(c)$.
>
> In mathematical notation, this is written as
>
> $$\lim_{x \to c} f(x) = f(c)$$
>
> In detail this means three conditions: first, f has to be defined at c. Second, the limit on the left hand side of that equation has to exist. Third, the value of this limit must equal $f(c)$. The function f is said to be continuous if it is *continuous* at every point of its domain. [7]

Note how the initial definition has only two conditions for continuity at a point with the condition of "function being defined at the point" stated outside of the "if" statement (D2). However, immediately afterwards the definition is again restated in terms of three conditions (D1). This second statement implicitly means that the violation of any of these three conditions would make the function discontinuous at that point. Also note that in the end, the definition for a continuous function is given so that it is consistent with D2, but not with D1.

How can this situation be problematic? According to D1 (the definition of continuity at a point adopted in the Wikipedia page), one condition under which a function can be discontinuous at a point is when the function is not defined at that point. This means that if a particular point is not in the domain of the function, then the function is discontinuous at that point. But despite this, if this function is continuous at the rest of the points which are in its domain, it is called a continuous function.

Consider the following function (also shown in Figure 3):

$$f(x) = \frac{x^2(x-3)}{(x=3)}$$

Note that, in the context of university first year calculus courses, the domain of a function is taken to be the largest set of real numbers for which the function is defined, unless specified otherwise. This function has a discontinuity at $x = 3$, but it is a continuous function!

Then come more inconsistencies. Under "classifications of discontinuities", Wikipedia states "If a function is not continuous at a point in its domain, one says that it has a *discontinuity* there" [8], hence swinging back to align with definition D2 and contradicting part of what is presented under "continuous functions" [7], where it implies that if a

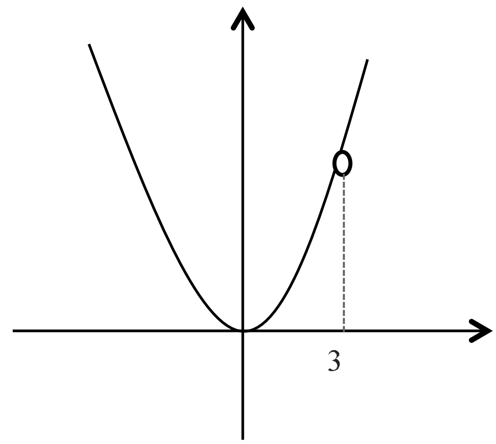


Figure 3. Graph of f(x).

function is not defined at a point it has a discontinuity there. It is clear, however, in this later description, that discontinuity happens at a point where the function is defined.

Could this case not be overlooked, since websites are not usually considered to be reliable sources? We cannot let students' frequent use of internet sources go unheeded. It needs to be borne in mind that the above presentation of continuity can plant conflicting ideas in a learner's mind. And these conflicts may only surface when dealing with a problem that evokes elements of both these definitions.

We invite the reader who wishes to reject Wikipedia as a reliable source to consider definitions from a calculus textbook. To define continuity at a point, Khuri (2003) starts with the epsilon-delta definition but quickly proceeds to elaborate it simply using the limit definition:

> Let f: $D \to R$, where $D \in R$, and let $a \in D$. Then $f(x)$ is continuous at $x=a$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that
>
> $$|f(x) - f(a)| < \varepsilon$$
>
> for all $x \in D$ for which $|x - a| < \delta$. (p. 67)

It is sufficient to look at the start of this definition to recognize that it parallels with D2, since prior to the condition for continuity it clearly states that a is in the domain (let $a \in$ D). This means if it is not, then we do not talk about continuity. Khuri does emphasize this issue:

> It is important here to note that in order for $f(x)$ to be continuous at $x = a$, it is necessary that it be defined at $x = a$ as well as at all other points inside a neighborhood $N_r(a)$ of the point a for some $r > 0$. (p. 67)

However, a few lines later, he states:

> Thus to show continuity of $f(x)$ at $x = a$, the following conditions must be verified:
>
> 1. $f(x)$ is defined at all points inside a neighborhood of the point a.
>
> 2. $f(x)$ has a limit from the left and a limit from the right as $x \to a$, and that these two limits are equal to L.

3. The value of $f(x)$ at $x = a$ is equal to L.

If any of the above conditions is violated, then $f(x)$ is said to be discontinuous at $x = a$. (p. 67)

These requirements are precisely isomorphic to D1, according to which a function would be discontinuous at a point if it is not defined at that point [9]. To be consistent with what is given in the initial definition, that is, to align with D2, the first condition should be stated as a pre-condition for continuity.

However, Khuri then swings back to D2 when presenting types of discontinuities:

There are two kinds of discontinuity.

Definition 3.4.2. A function f: D → R has a discontinuity of the first kind at $x = a$ if $f(a\text{-})$ and $f(a\text{+})$ exist, but at least one of them is different from $f(a)$. The function $f(x)$ has a discontinuity of the second kind at the same point if at least one of $f(a\text{-})$ and $f(a\text{+})$ does not exist. (p. 67)

The "first kind" of discontinuity occurs when the third condition is violated. And the "second kind" of discontinuity occurs when the second condition is violated. There is no discontinuity defined for the violation of the first condition which means that the two types of discontinuities are consistent with D2.

While the absence of definitions or inconsistent definitions can be observed in many textbooks, it is interesting to see Strang (1991) explicitly pointing out this disparity in definitions in his textbook: "it is amazing but true that the definition of 'continuous function' is still debated" (p. 87).

Situating the issue

We have considered two issues regarding the concept of continuity; namely, the absence of an explicit definition and inconsistent definitions. Where else do similar problems come up in the teaching and learning of mathematical concepts?

Absence of definitions

We do not claim that the absence of explicit definitions is *always* problematic. Not all mathematical concepts are taught via a definition. There are areas in mathematics in which we often see concepts being used and discussed without them being explicitly defined. This practice is common at the elementary and middle school levels. There are many concepts that are gradually developed through activities, experiences and exercises. This method closely resembles the method of concept formation via abstraction (Vygotsky, 1986). According to Vygotsky, concepts can be formed in two ways: through definition and through abstraction. Consider, for example, the concept of addition. It is developed via abstraction at the elementary school level and only those who study advanced mathematics are ever exposed to the formal definition of addition as a binary operation with certain properties. However, at high school and university, new concepts are introduced together with their formal definitions (*e.g.,* limit) and rigorous definitions are developed for concepts previously abstracted (*e.g.,* square).

Morgan (2005) discusses the implicit nature of some definitions. She shows that while no formal definition of dimension is given the participants in an elementary school mathematics class use their implicit definitions to form arguments about whether particular shapes fulfill the necessary conditions to be classified as two-dimensional. One of the things we have pointed out in this article is the possibility of learners deriving their own definition for a continuous function, but in a way that can be problematic.

Inconsistent definitions

Many mathematical concepts have more than one definition, but these definitions are equivalent. For instance the absolute value of a real number can be defined as,

$$|x| = \sqrt{x^2}$$

or as

$$|x| = \begin{cases} x; x \geq 0 \\ -x; x < 0 \end{cases}$$

Winicki-Landman & Leikin (2000) outline properties that must be satisfied by a definition and how one may choose a definition according to both mathematical and didactic considerations. One definition can be preferred over the other in a certain context, but the expectation is that different definitions used for the same concept are equivalent. But is this always the case?

A square can be defined in many different ways. Many of these definitions are in relation to a polygon (*e.g.,* a regular polygon with a 90° angle) or polygons are referred to via other quadrilaterals (*e.g.,* a rectangle with equal sides; a rhombus with a right angle). While the definitions mentioned above are equivalent, there is an ambiguity here whether a square includes its interior (Zazkis & Leikin, 2008). Indeed, there is substantial disagreement in the definition of a polygon [10]. A polygon is defined in some sources as a simple closed curve that consists of line segments. In other sources, it is defined as a closed plane figure bounded by straight line segments as its sides. While the former definition excludes the interior, the latter includes the region that the segments enclose. So whether a square includes or excludes its interior depends on the definition of polygon that is used in defining quadrilaterals. However this ambiguity does not create any problems because the properties of the square are not affected by the implied definition.

As another example, consider the following common definitions of an even number.

(a) An even number is an integer of the form $n = 2k$, where k is an integer.

(b) An even number is a natural number that is divisible by 2.

Is -6 an even number? It definitely is, according to (a). But according to (b) the property of evenness, as well as of divisibility in general, is defined with respect to natural numbers. As such, -6 is neither even nor odd, since it falls out of the scope of this definition. Also, zero is even according to (a) but is out of scope according to (b). The definition of odd and even numbers is context dependent. One defines even numbers on the domain of integers whereas the other defines even numbers on the domain of natural numbers. However, once the context is clarified, this disparity in the two definitions does not create any problems.

Similarly, the definition of quotient is context dependent. Consider the quotient in the division of 20 by 8. Is the quotient 2.5? Or it is just 2, with the remainder of 4? The context of the question guides the answer, as well as the definition of a quotient. Quotient is the result of division in the context of division of rational numbers. However, in the context of division of whole numbers, or division with reminder, a quotient is the whole part of this result. Again, once the context is clarified, the two definitions can be used with no confusion.

The inconsistency in the two definitions of continuity, however, is not a case of context dependence. Instead, the two definitions lead to contradictory conclusions within the same context. As illustrated in detail in this article, the problem is intensified when additional continuity-related concepts are defined using definitions that are inconsistent with each other simultaneously.

Hamdan (2008) writes about how the different sequencing of topics in textbooks leads to different definitions of logarithm. However, he shows the equivalence of these definitions and how any one of the narratives could be used as *the* definition and the other as a theorem. In another somewhat similar study to what we have presented in this article, Van Dormolen and Zaslavsky (2003) discuss different facets of definitions with regard to the case of periodic functions. They discuss different definitions that can be used for a periodic function and for the period of a function and point out how the same function can be periodic according to one definition and be non-periodic according to another definition. Emphasizing the importance of both consistency in related definitions and of choosing definitions when there is more than one available, they state that "whatever choice one makes, one has to be consistent" (p. 105). Both from a logical as well as an aesthetic standpoint, we believe that different definitions of the same concept should not lead to contradictions. And from a pedagogical standpoint, such situations should be avoided by choosing related definitions consistently so that any confusion that may be created due to inconsistent definitions is minimized.

Continuing problem of "continuity"? Implications and final remarks

While there are inconsistencies in the way the continuity of a function at a point is defined, there is both ambiguity and inconsistency in explaining, let alone defining, what a continuous function is. University instructors who teach calculus courses may choose to avoid the issue by not including functions that are not defined at a point in their presentation of the concept. For example, we observed a lecture in which the (dis)continuity of the following function was discussed:

$$f(x) = \begin{cases} \dfrac{1}{x-3}; x \neq 3 \\ 7; x = 3 \end{cases}$$

Disagreements in the mathematical community about definitions and concepts are not unusual in the history of mathematics. However, the issue is troublesome when it has pedagogical implications. We believe that textbooks and instructors should acknowledge the disparity and most importantly be consistent in the definitions that are used.

In Table 1 we show how D1 and D2, the two leading non-equivalent limit definitions for continuity at a point, may consistently build and derive the definition for a continuous function.

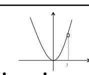
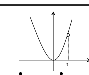
	D1	**D2**
Continuity at a point	A function f is said to be continuous at point c if, 1. $f(x)$ is defined at $x = c$ 2. $\lim_{x \to c} f(x)$ exists 3. $\lim_{x \to c} f(x)$ is equal to $f(c)$	Let c be a point in the domain of the function f. Then f is continuous at c if, $\lim_{x \to c} f(x) = f(c)$
Discontinuity at a point	If any of these three conditions fails, the function is *discontinuous* at $x = c$	f is discontinuous at point c in its domain if, $\lim_{x \to c} f(x) = f(c)$
Continuous function	A function is a continuous function if it is continuous at *every real number.*	A function is a continuous function if it is continuous at every point in *its domain.*
Example to illustrate the difference	 Function is not defined at 3. Therefore there is a discontinuity at 3. The function is not a continuous function because it has a discontinuity at 3.	 Function is not defined at 3. Therefore, the question of continuity or discontinuity does not arise at 3. The function is a continuous function because it is continuous at every point in its domain.
Effect on a point in a rational function where the function is not defined	Is called a "removable discontinuity"	Is called a "(removable) singularity" [11]

Table 1. Consistent definitions of continuity.

We do not claim that keeping to one of the two streams of definitions of continuity shown in Table 1 will make some of the confusions that can arise in learners disappear. In fact, the table is far from solving the problematic situation we have pointed out in the article. However, we believe that part of the solution lies in identifying and being aware of these problematic situations.

In their discussion on equivalent and non-equivalent definitions, Winicki-Landman and Leikin (2000) note:

> Each of the statements from the equivalence class of one of the definitions of a concept may be chosen as a definition. This choice is arbitrary and relies on both mathematical and didactic considerations. (p. 21)

In the case of continuity, the choice is between two streams of non-equivalent definitions and we suggest that it is important to develop mathematics teachers' awareness of the discrepancies discussed in this article so that they can make more informed choices. Furthermore, an examination of different definitions and their implications is a worthwhile activity for students, and can foster their understanding of the role of definitions in mathematics and their appreciation of precision and consistency.

Notes

[1] "Function (mathematics)", § "Alternative definition of a function", *Wikipedia*, retrieved 5 April 2013 from en.wikipedia.org/wiki/Function_%28mathematics%29#Alternative_definition_of_a_function

[2] See, for example, Stewart (2012), Tan (2011), Khuri (2003), Neuhauser (2010), Lial *et al.* (2011).

[3] In some sources (*e.g.,* Mathworld.Wolfram), this part of the definition is omitted. However, since definitions are "if and only if" statements, it is implied that the converse is true.

[4] See, for example, Stahl (2011), Binmore (1982), Bermant (1963), Strang (1991), Begle & Williams (1954).

[5] "HMC Mathematics online tutorial: Continuity", *Mathematics: Harvey Mudd College,* retrieved 5 April 2013 from www.math.hmc.edu/calculus/tutorials/continuity/

[6] Only some more advanced texts emphasize the necessity for the function to be defined at $x = c$ as well at all other points inside a neighborhood of the point c, in order for the function to be continuous at $x = c$ (*e.g.,* Khuri, 2003).

[7] "Continuous Functions", para. 6, *Wikipedia,* retrieved 5 April 2013, from en.wikipedia.org/wiki/Continuous_function

[8] "Classification of discontinuities", *Wikipedia,* retrieved 5 April 2013, from en.wikipedia.org/wiki/Classification_of_discontinuities

[9] In fact there is a further logical confusion here. The first condition has been stated as "$f(x)$ is defined at all points inside a neighborhood of the point a". If a discontinuity occurs by the violation of this condition (*i.e.,* when the negation of the statement happens), it means that a function would be discontinuous at a if "$f(x)$ is not defined at at least one point inside a neighborhood of the point a"!

[10] "Polygon", *MathWorld*, retrieved 5 April 2013 from mathworld.wolfram.com/Polygon.html

[11] For example, "If $f(x) \to \infty$ as $x \to x_0$ where $x_0 \in [a, b]$, then x_0 is said to be a singularity of $f(x)$" (Khuri, 2003, p. 225). Or "The term *removable discontinuity* is sometimes used in an abuse of terminology for cases in which the limits in both directions exist and are equal, while the function is "undefined" at the point. This use is abusive because continuity and discontinuity of a function are concepts defined only for points in the function's domain. A point not in the domain is properly named a removable singularity": see "Classification of discontinuities", Wikipedia, retrieved April 5, 2013, from en.wikipedia.org/wiki/Classification_of_discontinuities

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