A MODEL OF MATHEMATICAL LEARNING AND CARING RELATIONS

AMY HACKENBERG

Social interaction is basic to mathematics learning in the sense that student-teacher and student-student interactions form a major, though not comprehensive (*cf.* Confrey, 1995; Steffe, 1996), impetus for the learning of both teachers and students. Very broadly, I believe that any social interaction can trigger depletion and stimulation of the people involved. I define *depletion* as a feeling of being taxed in some way, usually accompanied by a decrease in energy or a diminishment of overall well-being. I take *stimulation* to mean a feeling of being excited or awake, usually accompanied by a boost in energy or a stronger sense of aliveness. In any social interaction these two factors may be negligible or pass unnoticed, and the dominant feeling may be one of 'evenness' or neutrality. However, in some social interactions the two factors fluctuate more obviously or one greatly outweighs the other.

In social interaction focused on learning, sustaining some level of depletion is often necessary for subsequent feelings of stimulation. Yet in mathematical interaction between teachers and students, depletion may dominate for a variety of reasons, such as students' attitudes toward mathematics (Hart and Allexsaht-Snider, 1996); students' considerable difficulty with the subject (RAND Mathematics Study Panel, 2002); teachers' instructional approa- ches; the structure of mathematics classes; and common beliefs about the nature of the subject (Henrion, 1997).

Although continuous feelings of stimulation are not possible, prolonged feelings of depletion may contribute to comments from both students and teachers that the other party 'doesn't care'. Some students will say that teachers do not care about students' lives or ways of thinking, which implies a lack of interest in or valuation of the students. These students do not feel *cared for* by their teachers. Some teachers will say that students do not care about – in the sense of are not engaged in – learning, a particular subject matter, or school. These teachers do not feel cared for *as teachers* because they do not experience their students' engagement and responsiveness to the activities they orchestrate and the questions they pose. These feelings may not correspond with the other party's intentions – teachers may care very much about their students, and students may be curious about and interested in learning, but teachers and students often fail to develop caring relations (Noddings, 1984, 2002). Such failures can interrupt or stunt learning for both students and teachers.

The purpose of this article is to describe a model of mathematical learning and caring relations, where *caring* is conceived of as work toward balancing the ongoing depletion and stimulation involved in student-teacher mathematical interaction. I consider reasons for building the model (*why*) to be satisfied by this brief introduction – to understand and account for mathematical learning and caring relations. I situate the model in school classrooms (*where*), and I intend the model to be about the learning of pre-adult students as well as their teachers (*who*), although my comments may have wider applicability.

Having briefly addressed *why*, *where*, and *who*, in the first two sections I focus on *what*: what I take to be acts of mathematical learning; what students and teachers learn; and what I define as mathematical caring relations. In the third section I focus on *how*: how students and teachers learn what they learn; and how they can engage in mathematical caring relations. I conclude the paper by describing some benefits of a model that holds learning and caring together.

Mathematical learning

A model of mathematical learning necessarily involves the learning of students *and their teachers* (Steffe and Wiegel, 1996), because a model addresses how teachers can learn to bring forth and sustain students' learning. However, what students and teachers learn is not identical. A central goal in model building is to delineate these realms of learning.

Acts of learning

In my view, acts of learning involve adaptive cognitive change, where cognition is the functioning of intelligence – endemic to human life (*cf.* Maturana and Varela, 1980). More specifically, I define acts of learning as adaptations – modifications or reorganizations – of ways of operating in the context of a person's interaction with their environment (*cf.* Piaget, 1970; von Glasersfeld, 1995) [1].

A conceptual scheme is a goal-directed way of operating that consists of a situation, activity, and result (Piaget, 1970; von Glasersfeld, 1995). To initiate the scheme, a situation must be perceived or recognized by a person as similar in some way to previous situations in which the person used the scheme. This perception or recognition is the result of assimilation (von Glasersfeld, 1995), the basis for construction – and modification – of schemes. The perceived situation then triggers the activity of the scheme, which may be mental or physical or both. The result of a scheme is an outgrowth of the activity, and the person generally anticipates that the result will be expected or satisfying in some way.

Perturbations

Modification or reorganization of a scheme may occur when a person's current schemes produce an unexpected result *e.g.*, the person does not achieve their intended goal. This 'disturbed' state of affairs is one example of a perturbation and is often accompanied by a sense of disappointment or surprise. A perturbation may also arise if a person intends

or wishes but is unable to initiate activity in a particular situation, a potentially frustrating state of affairs. A third example of a perturbation may occur if a person's activity in a situation – or result from the activity – seems incongruous to *another* person, who points out the incongruity. The extent to which the actor also comes to experience her own activity or result as incongruous determines the extent to which a perturbation happens.

As von Glasersfeld (1995) emphasizes, a person's "unobservable expectations" (p. 66) are instrumental in initiating a perturbation because what is crucial is the degree to which the unexpected result, inability to act, or incongruity 'matter' to a person at an intentional or unintentional level. This aspect of perturbations means they are not always consciously conflictive. An unexpected result, an inability to act, or someone else's sense of incongruity may remain largely unnoticed by a person and yet have some impact on that person's subsequent activity – perhaps in a vague sense of unease or a sort of heightened interest. Thus, even perturbations that are mostly outside of immediate awareness involve an affective aspect. As a person (consciously or unconsciously) eliminates perturbations, or equilibrates, the perturbation has the potential to trigger an act of learning.

Vertical and horizontal learning

Accommodations are acts of learning [2], and they can be categorized as primarily developmental or primarily functional. Developmental accommodations reconstitute a scheme on a new level and reorganize the scheme at that level (Steffe and Wiegel, 1994). Because the 'jump' in level of operating may be significant, developmental accommodations might be characterized as 'vertical' acts of learning. Functional accommodations of a scheme occur while using it (Steffe, 2002) and vary in the extent to which the scheme is modified. Some functional accommodations may produce vertical learning, but many yield more modest adjustments [3]. For this reason, many functional accommodations might be considered 'horizontal' acts of learning.

What do students learn?

Through both vertical and horizontal acts, students learn to order, comprehend, explain, and manage their experiential worlds so as to achieve a sense of prediction and control, as well as an ability to question, evaluate, and justify their ways of operating. This kind of knowledge is first-order knowledge (Steffe and Wiegel, 1996), sometimes referred to as *students' mathematics* (Steffe and Tzur, 1994). In building their mathematics, ideally students also learn to recognize their mathematical activity in relationship to conventional mathematical activity, at least to the degree that they can operate and communicate in the dominant mathematics culture [4].

What do teachers learn?

Teachers also increase their first-order knowledge of mathematics through teacher-student interactions. In the midst of these interactions, a teacher acts as a first-order observer because he or she,

> does not intentionally analyze the mental structures of the child relative to his or her own mental structures. (Steffe and Thompson, 2000a, p. 202)

However, building first-order knowledge is insufficient to describe teachers' learning if teachers want to learn how their students think and learn. To explain their experiences with students' activity, teachers learn to make models based on their first-order knowledge as well as their analysis of their students' first-order knowledge and learning (Steffe and Wiegel, 1996). Steffe and Tzur (1994) refer to these second-order models as the *mathematics of students*. When building the mathematics of their students, teachers act as second-order observers who:

> focus specifically on explaining the child's learning relative to [their] own purposes, intentions, and contributions to mathematical interactions. (Steffe and Thompson, 2000a, p. 202)

Since a teacher's second-order models rely on both the teacher's and students' first-order knowledge, these models can be thought of as co-constructed by the teacher and students in social interaction (Steffe and Wiegel, 1996). These models continually evolve in a dialectic relationship between model building and acting. That is, while building second-order models, teachers learn how to act on this knowledge so as to bring forth and engender modifications in students' ways of operating. While acting based on current models, teachers further refine their models. In addition, second-order models provide a point of contact between students' ways of operating and teachers' more conventional ways of operating. This connection facilitates students' participation in the dominant mathematics culture.

Mathematical caring relations

In Noddings' (1984, 2002) notion of caring relations, a teacher (a carer) orchestrates experiences in which a student (a cared-for) feels that her or his need for care in student-teacher interactions is satisfied. The student's response to the teacher's care completes the relation [5]. Thus caring is not simply a feeling, or a virtue of a teacher or student – caring is an orientation to create and participate in social interactions with certain cognitive and affective qualities. As a characteristic of *relations* between people, caring implies reciprocity. However, teachers and students participate differently in enacting (or failing to enact) caring relations. To articulate student-teacher interaction in my model, I adapt Noddings' work in formulating mathematical caring relations.

What is mathematical about caring?

Noddings (1984, 2001, 2002) believes that the caring of a teacher involves stretching students' worlds and working cooperatively with students so that they realize competence in those worlds. Therefore, she supports bringing forth mathematical competence in students as a major task of mathematics teachers. However, because at times Noddings prioritizes the development of caring people over learning in a subject area (such as mathematics, *cf.* 1993), she sometimes gives the impression of separating caring from learning. This separation may be warranted in discussing a

more general notion of caring.

I conceive of *mathematical* caring relations as inseparable from learning: mathematical caring relations occur in the context of aiming for mathematical acts of learning. Mathematics teachers may act as carers in general, but they start to act as mathematical carers when they work to harmonize themselves with and open new possibilities for students' mathematical thinking, while maintaining focus on students' feelings of depletion and stimulation that may accompany student-teacher interactions. That is, mathematical carers hold together their work toward mathematical learning and their work toward balancing stimulation and depletion in mathematical interactions with their students.

Mathematics teacher-as-carer

To harmonize with students, the teacher attempts to take on the students' mathematical realities – their ways of operating – as if they were the teacher's own. Doing so means that the teacher decenters, or sets their own mathematical ways of operating temporarily to the side in order to focus on students' ways of operating. Decentering does not imply that the teacher loses sight of their own mathematical reality, but instead makes space to see realities that may differ significantly from their own. The teacher tries to understand students' ways of operating by posing experientially real situations to the students, maintaining a playful orientation (Steffe and Wiegel, 1994), observing students' activity and results of their ways of operating, and asking questions about their work. In this process the teacher makes conjectures about schemes that students have built and tracks the stimulation and depletion students seem to experience as they act. Because of the attention to and valuation of students' ways of operating in these interactions, the teacher expects that students may experience at least some stimulation.

To open new possibilities for students' mathematical thinking, the teacher considers situations that might allow students to make modifications in their ways of operating – not necessarily so that students come to operate as the teacher does, but certainly in ways that students themselves may not imagine. Aware that feelings of depletion are endemic to processes of learning and may be necessary for significant stimulation, the teacher does not avoid posing situations that challenge students. However, the teacher also monitors and attempts to alleviate prolonged feelings of depletion (or excessive stimulation) that may harm engagement.

In this way, mathematical caring relations can be maintained even if specific acts of learning do not occur during any given interaction. For example, a student may not make an adaptation in a scheme for some time, but through the teacher's attention to stimulation and depletion (which may be outside of the student's awareness), the student may come to sustain engagement through perturbations – even through those that are consciously conflictive. Being able to sustain engagement through perturbation opens possibilities for further learning. In this way mathematical caring and learning are entwined.

Mathematics student-as-cared-for

Students who enter mathematical caring relations (and not all do) are likely to feel overtly recognized, to have a sense of being seen by the teacher. They may feel that they are being listened to, that their ideas are valued, and, perhaps, that they are understood. As a result, the students may experience stimulation – may feel energized or stronger in some way. These feelings may help students sustain or increase their engagement in mathematical activity, both that engendered by the teacher and by themselves. Through such engagement the students complete the caring relations.

A student who does not easily enter into mathematical caring relations may come to complete such relations through extended interaction with a mathematics teacher-as-carer. This comment is not meant to imply that teachers' work toward addressing their students' needs for care in mathematical interaction is simple or constant. The fluctuation of feelings of stimulation and depletion in social interaction aimed toward mathematical learning is sometimes subtle, often rapid, and not solely in the teacher's control. Furthermore, it is conceivable that one student's need for care may be satisfied in ways that are almost directly opposite to another's. For example, some students may need a great deal of significant challenges (*e.g.*, situations that are intended to provoke vertical learning) in order to feel mathematically cared for, while others may require such challenges much less frequently. Despite the difficulty in addressing different students' needs simultaneously, it is likely that the teacher's practice of decentering will facilitate students' overall sense of being cared for and thus engender students' participation in mathematical caring relations.

Reciprocity: students as carers, teachers as cared-fors

When a student's response to a teacher's mathematical care includes a "glow of well-being" (Noddings, 2002, p. 28) or renewed engagement in mathematical activity, the student 'cares back' for the teacher. This response is what the teacher receives:

> The student has his greatest effect on the relationship as the one cared-for. If he perceives the teacher's caring and responds to it, he is giving the teacher what she needs most to continue to care. (Noddings, 1984, p. 181)

The reciprocity is subtle. Students are not expected to care back for teachers in the same way that teachers care for students because, for example, students are not expected to monitor and assess the depletion and stimulation that teachers experience. What is required from the student to complete the caring relations is reception of and engagement with the teacher's care, which provides experiences of stimulation for teachers. In this way, teachers are cared for by their students and, according to Noddings, receive the "natural reward of teaching" (p. 182). [6]

Failures in caring relations

But what happens if students do not engage reciprocally in completing the relations? Well, they might not – caring relations can fail to be enacted due to the carer, cared-for, or other aspects of the situation. Just as teachers may teach but students may not learn, teachers may try to enact mathematical care and students may not respond in ways described here (*cf*. Noddings, 2001). Or at least, student responses to

a teacher's care may not be obvious to the teacher, or may include overt resistance to the teacher's intentions and actions. In those cases, a mathematics teacher-as-carer recognizes that she or he and the students are not in harmony in some way. The teacher tries to understand better what stimulates the students and what activities and interactions might draw the students toward engagement (note that such activities might be radically different from the teacher's initial ideas and might involve cessation of mathematical activity, or quite different mathematical activity, for some time.) The teacher, in reinitiating efforts to decenter, may also attempt to search with the students – to draw them into the search implicitly and sometimes explicitly – for mathematical activity and interaction that engages them.

How do mathematical learning and caring relations happen?

A discussion of how people come to receive and give care, how students learn to build first-order knowledge, and how teachers learn to build second-order models, indicates ways in which mathematical learning and mathematical caring relations mutually support each other. These ways include close links between acts of learning and caring at the level of schemes, as well as parallels between how teachers act as mathematical carers and as learners.

How do people come to receive care?

Noddings (2002) believes that our early memories of being cared for are foundational for continuing to act as a cared-for and for becoming a carer. If an individual grows up in a caring environment, or at least grows up with *someone* who successfully enacts caring relations with the individual, that individual will have an experience to draw upon in manifesting the responsiveness that completes caring relations. Noddings acknowledges that these people are the "lucky ones" (2002, p. 15). Many people grow up without having this need satisfied in any consistent way, which may interfere with their ability to respond as cared-fors (and carers). [7]

Receiving mathematical care is also based on previous experiences of caring relations with mathematics teachers-as-carers. In such experiences, teachers decentered to harmonize with and open possibilities for their students, students engaged enthusiastically in mathematical activity that was experientially real to them, and, overall, stimulation outweighed depletion in student-teacher interactions. Unfortunately, not all students (or teachers) have such experiences or have them consistently. Students who do may willingly enter into and maintain these kinds of mathematical interactions with teachers. As a result, these students also are likely to have many opportunities to build first-order mathematical knowledge.

How do students build first-order knowledge?

By acting in situations that are experientially real and involve them in mathematical activity with a playful orientation, students also become first-order knowers. In these situations students 'work' both consciously and unconsciously to eliminate perturbations [8]. Because perturbations involve an affective response to the unexpected, uncertain, or incongru-

ous, experiencing a perturbation can be accompanied by feelings of both depletion and stimulation.

Feeling depleted is a common reaction to a perturbation, especially if a person senses that they do not know what to do to eliminate it, or that activity to do so will be particularly onerous. Not immediately knowing what to do in a mathematical situation is common and necessary for learning, but protracted lack of knowing, or burdensome activity, or inability to act in a situation, can deplete a student. Depletion may manifest as simply a sense of fatigue, or more strongly as emotional states like dread, dislike, irritation, or anger. If a feeling of depletion is too great or extended for too long, a student may feel overwhelmed, which may impede engagement in mathematical activity either immediately or in the future.

However, perturbations can also provide stimulation in the form of a challenge, particularly if a person senses that they can meet or satisfy that challenge in some way, or that such activity itself will be enjoyable. Meeting a challenge in a mathematical situation may mean that the student assimilates the situation and knows how to act, even if they may not foresee a particular result. Stimulation may manifest as continued engagement in activity, or it may include more obvious emotional experiences of excitement, flow (Csikszentmihalyi, 1990), eager anticipation, and even joy. If a feeling of stimulation is sufficient, the student's subsequent interest in or curiosity about acting in the situation may prolong mathematical activity and open new opportunities for learning.

If, over time, feelings of stimulation outweigh feelings of depletion, the student may feel mathematically cared for. Thus, experiencing mathematical care is not necessarily tied to elimination of perturbations or resolution of problems, although both may be satisfying to students [9]. In a given mathematical situation, students may feel cared for in carrying out the activity of a scheme, in producing a result that makes sense to them, or in reflecting on that result alone or in discussion with others.

Although a teacher has no 'direct' control over how a student responds to a particular situation or perturbation, a teacher who works to enact mathematical caring relations will monitor students' feelings of depletion and stimulation and try to alleviate imbalances that may impede engagement. The teacher does so in order to sustain opportunities for students' acts of learning as well as the teacher's own learning. But there are additional reasons that teachers practice mathematical care.

How do people come to give care?

Noddings (2002) notes that the enactment of caring is variable, dependent on situations and conditions. She defines *natural caring* as the state in which carers respond because they want to. In these cases, performing an act of care for another is in agreement with "ordinary life" (p. 13) and there is no inner conflict. The caring expresses, or is in harmony with, the carer's inclination and is not a recognition of duty.

In contrast, *ethical caring* (Noddings, 2002) occurs when a carer meets internal resistance to a belief that they should perform a caring act – the carer does not want to respond.

In these situations carers can ask themselves how they would behave if conditions were different – if the person who needs or is requesting care was not irksome, if as carers they did not feel so tired, if the need for care did not seem so overwhelming or the act of caring too onerous. In this way carers may draw upon their ethical ideal – the sense of themselves as caring persons – that is formed based on memories of caring and being cared for. Maintaining this ethical ideal can be the impetus for ethical caring. In turn,

> [e]thical caring is always aimed at establishing, restoring, or enhancing the kind of relation in which we respond freely because we want to do so. (p. 14)

In other words, the preferred state for a carer is natural caring, and ethical caring can be used to re-establish it. [10]

To give mathematical care involves both types of caring. Teachers-as-mathematical carers respond to their students' needs for care because teachers want to – that is part of why they have become teachers. They are intrigued by the experiences of their students and how those experiences change in student-student and teacher-student interactions. However, teachers also regularly meet resistance to enacting mathematical care. For example, students may seem particularly scattered or insufficiently engaged despite the teacher's decentering efforts. Teachers are consistently tired and busy, and the act of trying to harmonize with and open possibilities for their students' ways of operating is a significantly complex job. Teachers may persist as mathematical carers in these situations because of their own memories and images of being mathematically cared for and of caring mathematically; because of beliefs that through acting out of duty they will return to a more 'natural' state of mathematical caring; or even because they see caring through resistance as fully in the scope of enacting mathematical care.

How do teachers build second-order models?

To build second-order models, teachers must cultivate awareness of how they and students interact, and of the consequences of different choices for and habitual ways of interacting (Steffe, 1996). Such consequences include increasing or decreasing students' feelings of stimulation and depletion as well as opening or foreclosing possibilities for students' acts of learning. Thus building second-order models is tied to enacting mathematical care.

Cultivating awareness of student-teacher interactions involves teachers in decentering, which requires listening carefully to (Confrey, 1998; Davis, 1997) and closely observing students' ways of operating. The teacher's main goal in listening and observing is not to confirm their own mathematical thinking but to make images of and conjectures about *students' mathematics*. The teacher does not expect students to think like the teacher does. The teacher does expect to be surprised (in the sense of delighted, stimulated, and challenged) by how students think. In this sense the teacher works to harmonize with students, an aspect of enacting mathematical care.

However, to build second-order models teachers also must act. Based on their images of and conjectures about students' ways of operating, teachers pose situations to pro-voke vertical and horizontal acts of learning. Such situations can expand students' mathematical realities in ways that students have likely not envisioned, which can be stimulating for students and is another aspect of enacting mathematical care. But because such work can also be taxing, student-teacher interactions focused solely on vertical and horizontal learning have the potential to sustain depletion. So, students and teachers also spend time on situations that are not intended to bring forth specific acts of learning, but that can allow students to build confidence – and pleasure! – in their ways of operating. Through acting in these situations, students may feel successful, believe that their ways of operating are valued and useful, and therefore experience stimulation. As teachers continue to listen, observe, and reflect on the consequences of these interactions with students, teachers can further refine their models.

Thus, teachers learn to build second-order models of students' mathematics through the same processes by which they enact mathematical care: decentering to harmonize with and open possibilities for students while maintaining focus on feelings of depletion and stimulation in interaction. In turn, these models inform how teachers can enact mathematical care and provoke mathematical learning. So, a second-order model is a dynamic 'mechanism' by which mathe- matical caring relations and mathematical learning can be engendered.

Conclusions: benefits of a model of learning and caring held together

A central proposal of this paper is that attention to mathematical caring relations in the context of mathematical learning, teaching, and research will enhance students' engagement in mathematical activity, even through perturbations that are consciously conflictive. This first potential benefit has significant ramifications, since being able to sustain engagement through feelings of depletion opens opportunities for learning that may be foreclosed if the student withdraws from activity or engagement. In this way, the model is intended not just to bring forth results of acts of learning – more powerful schemes – but also more resilient affective states.

A second benefit of the model is that explicit articulation of mathematical caring relations may facilitate articulation of experiences of learning and teaching. That is, although many researchers offer analyses of learning and teaching, fewer capture what experiences of learning are like for learners, what experiences of teaching are like for teachers, and what experiences of student-teacher interaction are like for both. One reason for this absence is that verbalizing these experiences is very difficult. Another reason is that such experiences are often taken as an unquestioned ground for knowledge and research, which is problematic because experiences are what must be accounted for and explained (Scott, 1992). Finer articulation of learning and teaching experiences can help illuminate how they are taken as given and can facilitate analyses and accounts of them. Thus, mathematical caring relations may allow researchers and classroom teachers to 'converse' about experiential foundations that are often left implicit in research on learning and teaching mathematics.

Third, holding learning and caring together disrupts the traditional and harmful separation of the production of knowledge from the production of people (Rose, 1994) – of intellectual activity from emotional, embodied states. The usual division, which is often marked by gender, class, and race, can perpetuate inequities in education and society by associating abstract reasoning with some groups of people (usually those with considerable power) and by linking embodied maintenance of people and environments with other groups of people (usually those with less power). Some believe that such a division is at the heart of perpetuating not just inequity but severe harm to the future of the planet and human race (*e.g.*, Walkerdine, 1994). This model posits the production of knowledge and people as necessarily conjoined, argues that intellect and affect are unavoidably entwined (*cf.* McLeod, 1992), and offers a vision for addressing both in mathematical interaction.

Finally, I hope that this model contributes to the ongoing debate about the role of social interaction in learning. My model articulates the role of social interaction as more than intellectual or cognitive but as necessarily affective, highlighting student-teacher interaction as integral to but not identical to mathematical learning. In particular, the model explicates how student-teacher interaction can affect engagement with mathematical activity that is essential for acts of learning to occur. In addition, the model points toward the necessary involvement of ethical issues in considering the role of social interaction in learning. The emphasis on care and concomitant notions of ethics of care (*cf.* Gilligan, 1982; Noddings, 2002) is one aspect that, with further articulation, may respond to calls for models of both social interaction and ethics compatible with constructivist theories of knowing (Lewin, 2000; von Glasersfeld, 2000).

Acknowledgements

Notes

[1] Interactions include those with people, non-human entities, as well as interactions of constructs within a person (*cf.* Steffe and Wiegel, 1996). Note that a person's environment is everything that an observer sees that is *not* the person (*cf.* von Glasersfeld, 1995).

[2] However, I regard *generalizing* assimilations (one type of assimilation) as modifications of schemes, and therefore as acts of learning (*cf.* Steffe and Wiegel, 1994).

[3] Developmental accommodations include reorganizations of schemes based on prior functional accommodations. Thus they occur as a result of *learning* (Steffe and Thompson, 2000b), not just maturation.

[4] I use 'dominant mathematics culture' to refer to traditional ways of knowing, representing, and communicating mathematics, including both verbal communication and written notation.

[5] Noddings uses the word 'completes' to indicate that without the involvement of *both* carer and cared-for, caring relations cannot occur. This emphasis contrasts with the notion of caring as a characteristic of an individual, who can care even if the one in her care does not feel cared for. The word is not meant to imply that in on-going interaction caring relations are ever fully finished.

[6] An expanded note on who cares is important given the controversy around the notion of caring in feminist theory. Since women have traditionally been care-givers and women teach students from ages four to eighteen in far greater numbers than men, promoting teachers-as-carers may maintain stereotypes that caring is simply what women 'naturally' do. Thus, an emphasis on caring in teaching and learning may perpetuate exploitation of women. I respond to this criticism similarly to Noddings (2001, 2002). To say that female teachers enact caring with their students more than male teachers is absurd, unknowable, and not my project. However, I don't want to deny the care-giving history of women. Most important, I do not wish to throw out notions of care simply because they have been associated with women. Instead, I prefer to explicate the 'work' associated with caring (*cf.* Rose, 1994) so that rather than be regarded as solely 'natural,' caring might be seen as valuable learned behavior that is in the province of all people. In this sense I depart somewhat from Noddings' reliance on 'natural' with regard to caring.

[7] Noddings notes that even people who have few memories of being cared for are sometimes able to transcend this impoverishment and enact themselves as cared-fors (and carers).

[8] As already discussed, perturbations include, but are much more general than, conscious cognitive conflict.

[9] For students, experiences of building first-order knowledge may be that of solving problems, although accommodations and assimilations explain more than problem solving (*cf.* Steffe and Wiegel, 1996).

[10] However, just because enacting ethical caring can maintain caring relations does not mean that a carer always must – or can – enact it! Noddings (2002) acknowledges that:

> Care theory does not attempt to develop a model of moral education that can produce people who will behave virtuously no matter how bad the world that surrounds them. (p. 9)

That is, for a variety of reasons, sometimes a carer cannot – or chooses not – to enact care. Setting such boundaries may actually maintain the carer who, because of their orientation, will likely return to initiating, maintaining, and enhancing caring relations in the future. This caveat also holds for enacting mathematical care.

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These notes and references follow on from page 38 of the article "Using warranted implications to understand and validate proofs" by Keith Weber and Lara Alcock that starts on page 34 (ed.)

Notes

[1] Utterances such as "umm", stutters, and repeated words were removed from the transcripts that are quoted to increase their readability. The text […] denotes that short segments of the transcript were deleted.

[2] Perhaps one could argue that the first implication is more acceptable than the second because its proof is more obvious. However, this would require the reader to infer the author's intentions for why this would be easier to prove. Such an action, to us, would closely correspond to inferring a warrant.

[3] Note that this is not due to carelessness on the part of the author. It is widely acknowledged that proofs would be impossibly long if each logical detail was included (*cf.* Davis and Hersh, 1981).

[4] The reader instead should be concerned with whether the assumptions used are part of a legitimate proof structure or framework (*cf.* Selden and Selden, 1995).

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