DISCUSSING BEAUTY IN MATHEMATICS AND IN ART

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David: I have just finished reading the article by Silver and Metzger (1989) you recommended to me I found it interesting in a number of ways. I am certainly sympathetic to the idea that there is an important (I'd say essential) emotional element in any reasoning or decision-making process, but I come at the idea from a different theoretical frame – Damasio's (1994) neuroscience mixed with Maturana's (1988) biology of cognition. That means I don't see emotion, aesthetics, metacognition and cognition as a set of interrelated processes, but rather aspects or facets of one process. But the basic idea, the importance of looking at these things, certainly strikes a chord, and I'd love to talk more about it with you

Before that though, I'd really like to know what you think about the ideas at the end of the article about education. We know that what is beautiful to a mathematician is not necessarily beautiful to a student, and Silver and Metzger rightly point out that this aspect of learning mathematics is worth more discussion and research As you are an artist who must have explored learning to appreciate beauty in other domains, I wonder if you have some insight into the process of learning to see (or hear or .) beauty Do we need mathematical beauty appreciation classes or something?

Eva: To me, seeing (or hearing or ....) beauty is inherently related to creativity in that it is an integral part of it How can I create something (beautiful), without having an appreciation for the concept. But to 'teach' creativity might be the wrong way to think about it. I would prefer to 'give opportunities for creativity to develop'. I rather think that everyone is born with some creativity, after all, most children like to draw, but it needs to be encouraged so it can flourish, and it is not allowed to flourish often in mathematics classes. One objection raised by teachers of those classes is that their students might not 'get it right'. But that communicates a lack of trust in learners and trust is exactly what I think is needed to sustain creativity (and the rigour it requires in the case of mathematics)

In mathematics there is the convenience that you can 'verify' the results, for instance, by using an alternate method, as opposed to art, where the 'test' seems to be public, or 'expert', opinion, that is, that of an external agent. Perhaps this links back to the historical context of proof, since 'proving' that 2 + 2 = 4 by example might have been sufficient for the early scribes, but certainly not to the formalists of the twentieth century! In this sense, there is an 'expert' verification process.

I do think that beauty is linked to creativity if only through the creativity of looking (or listening, or .). In my view something human-made is beautiful if it demonstrates creativity, yet rigour.

David: I am glad proof has come up, as that is central to my interest in emotions. One of the questions I have been thinking about is "How do we know when a proof is a proof?", and part of the answer to that is that, to a suitably trained mind (an expert?), it feels like a proof. That feeling can take a few forms. We might feel certainty where there was doubt before, or understanding where before there was confusion, or a sort of "Aha!" At least some of these feelings are also associated with whatever it means when a mathematician says of a proof "It's beautiful!", so aesthetics comes in here somehow.

Eva: It is interesting that you bring up the question of when a proof is a proof, especially in its link to the 'trained mind' aspect. If we use the analogy of mathematics as an art form, the equivalent case might be when is an art work finished, that is, when even a single added line/stroke/blot/sound would be superfluous and potentially destructive.

This, of course, is from the point of view of the maker of art/mathematics. On the external, appreciation-of-the-finished-work side, even discussing beauty, or appreciation, in art (or mathematics) is problematic as well. As you said, training, it could be argued, is part of the requirement for appreciation In the art context, for example, it certainly helps to understand the impressionists (for example) better, if there is a sense of the context of the time, including the fact that they were rebels who redefined the purpose of art and what it represented. In addition, it is also interesting to contextualise art in terms of the available technology at the time. In the case in point, the development of photography meant that exact records such as realistic portraits did not have to come from the brush of trained painters any longer, thereby leaving them to redefine their purpose and explore more subjective perspectives

Perhaps the analogy with mathematics is useful The validity of proofs also has a history. If we consider that mathematics was developed initially for accounting purposes and proofs were not an issue until about 600 BC (Davis and Hersh, 1981, p. 147), it might be useful to consider the historical context of a mathematical proof, or indeed of the theorem it proves. In today's context, the issue of the validity of computer proofs (notably of the four-colour theorem of maps) comes to mind. This is not to say that teaching the history of mathematics will resolve this. I would say though that considering the cultural context of the prover as well as the observer would be useful, just as it is in art

David: Your mention of history reminded me of an analysis class I sat in on a while ago. The professor took a lot of time and great care over the technical details of the proof, without ever mentioning why (historically) it is important to prove the theorem (which was something basic about Dedekind cuts being equivalent to the real numbers). When I asked the professor why he hadn't talked about the historical importance of the theorem he was unaware it had any. This astonished me, as I can't imagine teaching something if I didn't know why it mattered.

Perhaps historical importance is a separate issue from aesthetics. But perhaps it isn't. Euclid's proof of the infinitude of primes is usually thought to be beautiful (it is the very first proof in the book *Proofs from THE BOOK* (Aigner and Ziegler, 2004), a compilation of allegedly beautiful proofs) But the proof presented in modern texts is very different from Euclid's proof. His includes a geometric figure and uses a generic example; modern treatments are always purely algebraic and general. Perhaps the beauty is structural, not presentational? Is there any equivalent in visual arts to this kind of not-skin-deep beauty? I suppose so, especially in abstract art, which is why you need training to appreciate it.

Eva: Your distinction between skin-deep and more deeply rooted beauty in proofs could have an analogue in art, but the problem can be seen as more complex: the success of a work of art lies not simply in its aesthetic appeal, but, especially since the twentieth century, in what it is expressing. In fact, often in contemporary art, beauty is not the aim. Rather, the art aims at pointing out flaws or horrors, or even making fun. It becomes a language to express thoughts, critiques, approval or disapproval. In such instances, a work of art is then rated for how 'successfully' it accomplishes this aim. This, I think is perhaps closer to the 'beauty' we are talking about. After all, we find a proof beautiful (I think), when we find that it accomplishes its aim in a spare and elegant way, without anything superfluous.

Calling it elegant, rather than beautiful, is not simply a question of semantics, but rather it shifts the focus from an aesthetic one to an evaluation based on rational criteria, which can be more objectively accounted for than personal preference. This also connects to ideas in art education like proportions, composition, colour theory, which are then tools that artists can use to manipulate the perception of their finished work. Conversely, in mathematics, we use symbolic notation, diagrams and other conventions to make a proof appear short and elegant. Try to write any modern proof without those and you will need pages and pages of text (see, for instance, Euclid's *Elements* before they were re-written with symbols).

David: Your point about the importance in art of expressing an idea or position is very interesting. The roles of beauty and ideas in mathematics and art seem to be historically different. In mathematics, a theorem and a proof must express an idea. To be beautiful is desirable but optional. In art (at least in the past) a work must be beautiful; expressing an idea is optional. This suggests modern and contemporary art are becoming more mathematical. Perhaps art is yet another victim of *Descartes' dream?* (Davis and Hersh, 1986)

Your shift from "beautiful" to "elegant" reminded me of Rota's shift from "beautiful" to "enlightening" in his article *The phenomenology of mathematical beauty* (1997). He claims this shift, like yours, moves from aesthetic criteria to objective criteria, "The property of being enlightening is objectively attributed to certain mathematical statements and denied to others." (p. 181) I am not so sure those criteria (either his or yours) are really objective. I can certainly think of ugly proofs that are short and written using symbolic notation. Their brevity gets in the way of their accessibility, and hence their beauty/elegance (at least for me). Rota's property of being enlightening also seems problematic to me. Not everyone finds the same proof enlightening. Rota isn't very clear about how this is objective. Any ideas?

Eva: I don't think we can say that beauty used to be the goal of art but is no longer. It might be more correct that we, from the contemporary perspective, judge 'older' art for its beauty. But to the Egyptians, their monuments were not erected to decorate the landscape. And the renaissance portraits were not done for beauty's sake either. Indeed, their purpose was often pragmatic (appeasing the gods, representing a moral lesson, flattering the vanity of the rich and powerful, expressing their power, ...) Beauty, or what was considered beautiful, was indeed used as one more tool to express or explore something else. Perhaps we make the same 'mistake' when we consider proofs?

This brings us back to the 'historical context' issue, really, as it is not enough to look at an art work (or indeed a proof) from our own twenty-first century perspective. It is necessary, to really understand the piece, to consider the context that is contemporary to its creation/discovery/development, and particularly the philosophical/cultural/social perspective of the time. This also negates any kind of 'rational or objective' attribution, as you suggested. Instead, it might be more useful to consider the characterisations that exist at various times, and relate them to different proofs. Taking the example you cited earlier, of Euclid's proof, it might be interesting to examine its expression in various documents since its discovery, and perhaps consider what the criteria were at those various points in history. As for the 'rational' criteria, they are merely criteria of characterisation, and are relative at best.

You seem to have your own criteria for a 'beautiful proof'. In the absence of a criterion for brevity, which is certainly not a guarantee of elegance, what do you see? In any case, it is probably not so much that we could generate a list of criteria, then rate a given proof and calculate its elegance/beauty index. It is probably more relevant to discuss the considerations that evoke beauty/elegance when we examine a proof in general.

David: I did a little study of what proofs mathematics educators prefer (Reid, 2005). At present, my thinking about how I know a proof is beautiful (or a proof) is guided by the work of Damasio and Maturana. Maturana (1988) sees mathematics and all other "domains of explanation" as being practices of those who share a certain "emotional orientation". In the cases we are considering, the people we consider to be mathematicians are precisely the people who possess a mathematical emotional orientation, which includes implicit criteria for what a proof is and what a beautiful proof is. What those criteria are remains (necessarily) implicit (because even if you could specify them you would have to use other implicit criteria for judging criteria to do so). A beautiful proof is a proof that seems beautiful to a person who makes similar judgments to others who judge proofs similarly. This may seem circular, and it is in fact recursive, but Maturana's basic assumption is that life is like that and a more lineal story would miss important aspects of what life is like.

Damasio (1994) provides evidence (for those with a scientific emotional orientation) that human decision making in general (not just decisions about the beauty of proofs or art) has an emotional basis. He postulates "somatic markers" as the neurological mechanism that makes decision-making possible. Explicit rational criteria would be quickly overwhelmed by the complexity of day-to-day decision making. Pre-conscious somatic markers limit the options that come to mind and tag them with values to make deciding easy for the conscious mind. I see an emotional orientation as a set (a constellation) of somatic markers.

Perhaps there is a somatic marker that values brevity positively, but another one that values transparency positively. A brief, transparent proof would be more likely to be perceived as elegant, than a long opaque proof would be. But all sorts of other somatic markers would come into play as well (including, as Hanna, 1983, pointed out, opinions of the author of the proof and the field of mathematics involved).

Eva: Your discussion of the implicit, and possibly circular nature of these (emotional) judgments about mathematical proofs is interesting, particularly in light of the earlier, analogous discussion about the 'education' required to judge art work. You discuss the existence of "a person who makes similar judgments to others who judge [proofs] similarly". Is this not what we attempt to instil in our students, however implicitly, when we educate? Perhaps the issue lies in the transient nature of what is considered a (beautiful) proof, historically speaking, and as your earlier example about Dedekind cuts showed. After all, why would mathematicians look for new proofs, if the only concern were to have one, in order to 'tick' the theorem as proven?

In social studies, when a particular historical event is studied, its social context is generally considered an essential component of the topic. Perhaps this should be the case more often in mathematics. It may even alleviate some students' disinterest in the subject. Conversely, perhaps, if the actual need to prove were given more attention in the mathematics classroom, not only in terms of 'having covered the proof' of a theorem, but as an experience of questioning mathematics, then the purpose of proving could be more relevant to sceptics.

On another topic, in the earlier discussion on the distinction between beautiful versus elegant/enlightening, the use of 'rational' may be less a suggestion that the criteria can be reified and categorised systematically (using Kelly's, 1955, *Personal construct theory*, for example), and more that they can be understood through an examination of the discourses taking place within a relevant community of practice (Lave and Wenger, 1991). Indeed, the phenomenon of beauty in mathematical proofs is perhaps more a matter of *verstehen* than of *erklären* (Dilthey, 1976), particularly since mathematical knowledge is now more and more thought to be 'socially constructed'. This suggests to me that there is a benefit in letting "peripheral participants" (Lave and Wenger, 1991) take a more active role in such discourse, if only to increase their personal stake in said discourse. And for that, it seems necessary, to my mind, to find a way to develop a context where the learners can be trusted to 'get it right'. This brings us back, again, to an examination of the interaction between creativity (in mathematics) and mathematical rigour (which helps to 'get it right', or at least to judge a proof or result).

David: Your final comment makes me think of Bill Byers's book *How mathematicians think* (2007) which I have only begun to read, but he focuses on the balance between creativity and rigour in mathematics. If I understand you correctly, you see the key to appreciating the beauty of mathematics, and to understanding proof, as doing mathematics in a legitimate way. Analogously, to appreciate paintings you would suggest a course in painting rather than in art history. Or perhaps a method of teaching art history that included learning the techniques and styles of the paintings being studied, as well as their dates and social contexts. This would be very different from the way mathematics is taught now, and perhaps different from the way art is taught as well. It is a thought-provoking idea.

Eva: I agree that the idea of comparing mathematics and our social perception of its nature with that which we hold of art is a promising direction, especially if we consider the ideas that mathematics has a historical context, just as art does, and that our appreciation of either might use similar mechanisms. This could be the subject of deeper research, particularly, for me, with respect to the connection between making and appreciating, and the role of creativity in both.

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