TOWARD A MORE POWER-FULL SCHOOL MATHEMATICS

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One of the most common criticisms of contemporary school mathematics is that its contents are increasingly out of step with the times. The curriculum, it is argued, comprises many competencies that have become all but useless, while it ignores a host of skills and concepts that have emerged as indispensible. Much of the problem lies with a system that is prone to accumulation and that cannot jettison its history. Programs of study have thus become not-always-coherent mixes of topics drawn from ancient traditions, competencies imagined to constitute a necessary skill set for a citizen of the modern (read: industrially based, consumption-driven) world, necessary preparations for postsecondary study, and ragtag collections of other topics that have been seen to add some pragmatic value at one time or another over the past few centuries. Somewhat ironically, a domain that has not been particularly influential in these evolutions is mathematics itself. As a result, few current curricula have any substantial content that is reflective of developments in mathematics over the past few centuries.

The issue of accelerating irrelevance is particularly apparent around the notion of "basics" or, more obviously, "basic operations." This phrase is almost universally understood as a reference to a four-member set consisting of addition, subtraction, multiplication, and division. In spite of years of efforts to encourage more nuanced interpretations of the discipline, these operations are the mainstays of mathematics for most. Indeed, the "basic operations" and "mathematics" might be argued to be coterminous for a significant portion of the population. This correspondence is evident in such imperatives as, "Do the math!" and in questions like, "Why teach math now that everyone is carrying a calculator?" The unfortunate metonymy is no doubt held in place by the fact that, for the vast majority, most of the time given to mathematics in the elementary school years is/was devoted to these operations.

But in what ways are these operations basic? Certainly not in the sense of irreducible fundaments, starting premises, or irrefutable axioms, given the manner in which most school curricula initially define all four in terms of prior operations: addition in terms of counting, subtraction and multiplication in terms of addition, and division in terms of multiplication and subtraction. It seems to be that what these operations are basic *to* is not mathematics or mathematical understanding, but the needs of a minimally numerate human in an industrialized society. In other words, the meaning of *basic* at work here is not the one that is invoked with its cognate *basis*, which is suggestive of a stable foundation, defensible starting place, or universal essential. Rather, a more temporally and contextually specific meaning seems to be at play, one that is perhaps more closely fitted to the origins of the word. (Basic derives from the Greek *bainein*, "to step.") Grumet (1995) made this point by critiquing the habit of freezing competencies that are situationally specific into elements that are treated as eternal and universal. As she noted, what is essential is not a concept itself, but "the relation of […] histories of human action and interpretation to the lives of the children studying them" (p. 19). Within school mathematics, the basics are basic because of their necessity to a group of people at a particular time, not because of their role within a body of knowledge.

What, then, are the basics of school arithmetic in this current historical moment? It is difficult to argue against the continuing relevance of the current roster of addition, subtraction, multiplication, and division. Clearly these are of vital importance in the number-dense world we have co-created. However, as I develop in this article, they may be inadequate for the rapidly changing world that we now inhabit. To that end, I present a preliminary argument for an elaboration of the basic operations to include exponentiation and logarithms—hastening to add that this expansion is not at all about extending the list of procedural skills that children might be expected to master. Rather, I share with many the conviction that we must trouble the popular tendency to conflate mathematics and computation.

Coupling quantity and form

At a recent mathematics education research conference, I attended a symposium on the topic of mobile apps intended to support mathematics learning [1]. During the session, I reviewed a few iPad-based games that dealt with arithmetic. More accurately, they dealt with computational competence. That was no surprise. But something happened that did catch me off guard while playing. I made an adding error during one of my turns and the app's response was not, "Check your calculation," nor even, "Check your arithmetic." It was an all-too-familiar, "Check your math."

No doubt a major reason for my surprise was the fact that, in the next session, I was scheduled to be a discussant in a symposium that dealt with early-years spatial reasoning. Part of the backdrop for that panel discussion was a call for a greater emphasis on spatiality in school mathematics, as for example expressed in NCTM's (2006) *Curriculum Focal Points.* I find this move to be particularly encouraging, and not merely because it represents a serious attempt to break away from the common perception that "math = computation." As part of my discussant remarks, I used a slide of the 35 branches of mathematics listed in the Wikipedia "Mathematics" entry, each of which is assigned an iconic image on the site [2]. For me, this assemblage of images afforded a powerful demonstration of the manner in which *quantity* and *form*—or, if you will, numerical reasoning and spatial reasoning—are so intricately intertwined in most areas of mathematical inquiry. The move to amplify the role of spatiality in programs of study, then, seems to hold some promise for interrupting popular perception as it, potentially, pulls school mathematics toward a stronger semblance of its parent discipline. Indeed, the suggestion has been made more than once that it was Descartes' masterstroke of uniting number and form (through a coordinate grid) that marked the emergence of mathematics as a coherent and multifaceted domain half a millennium ago (see Davis, 1996, for a review).

It is partially in the spirit of supporting a more nuanced appreciation of the parent discipline that the proposal to expand the current list of basic operations is made. My eagerness to lay out this point is occasioned by a consistent response from educators and mathematicians to the suggestion that exponentiation should be considered as basic. More than one has cautioned me that the calculations quickly become too difficult, the concepts too complex, or the quantities too incomprehensible.

In response, and to re-emphasize, the intention here is not to include another site for computation. It is to seek a more meaningful, relevant mathematics, one that pulls together number and form as it offers a useful tool for interpreting and predicting the world. To that end, most of what I suggest below is concerned with developing a *feel* for exponentiation, specifically, and a sense of what mathematics is all about, more generally. It is also to support the development of better quantity sense (Wagner & Davis, 2010) among students. To invoke an old nugget, I do not contest the importance of knowing *how* to calculate, but that competence pales beside the need to know *what* is being calculated. Further, I am generally taken aback when, for example, a politician glosses over costs that overrun by factors of two or three, and I am nothing short of appalled when that same politician confuses billions and trillions. These are not the same sort of error. One is multiplicative; the other is exponential.

Encountering exponentiation—superpersonal

Several years ago, I encountered two statements that were roughly equivalent in mathematical terms and worlds apart in experiential terms. One was a remark during an election campaign in which a candidate sought to assure my fellow Albertans about the stability of our economy, pointing out that its basis in oil and other energy resources was in little danger from renewable sources. Less that 1% of the world's needs are currently being met by greener sources, he emphasized. A continuing market for our goods is clearly assured. By coincidence, I happened to watch one of Ray Kurzweil's TED talks [3] on the same afternoon. Touching on the same topic, Kurzweil pointed out that civilization is currently a mere eight doublings [4] away from meeting all its energy needs through renewable sources. In other words, by using a ratio to describe the situation, the Alberta politician appears to have been thinking in the space of multiplication, where the predictability of linear growth afforded confidence in a stable future. Kurzweil's thinking, by contrast, was in a more exponential mode. It painted a very different picture.

Kurzweil is among the best known of current futurists, and he is one of the leading exponents of thinking in terms of powers rather than factors. His publications (*e.g.,* 2005) are rife with logarithmic and exponential curves. To provide a sense of the sorts of commentaries on change that he offers, on the matter of computer processing technology he suggests that by 2029 (relative to 2006):

> we will have two-to-the-25th-power greater price performance, capacity and bandwidth of these technologies, which is pretty phenomenal. It'll be millions of times more powerful than it is today. We'll have completed the reverse engineering of the human brain, $1,000 of computing will be far more powerful than the human brain in terms of basic raw capacity. Computers will combine the subtle pan-recognition powers of human intelligence with ways in which machines are already superior, in terms of doing analytic thinking, remembering billions of facts accurately. [5]

Note that, insofar as such statements pertain to my thesis in this article, the point is not the accuracy of Kurzweil's prediction; it is the mode of thinking employed to make that prediction. He is likely wrong and, within the frame of exponentiation, perhaps spectacularly so. But that is part of the issue. The natures of speculation and error change when thinking in exponential terms.

It is interesting in itself that this particular insight—on the volatility of prediction—has become rather commonplace, no doubt in part because of the way the "Butterfly Effect" has captured the collective imagination. But even here, the actual mechanism at work is obscured through the manner in which the phenomenon is described. It is most often stated in terms of a nonlinear system's sensitivity to initial conditions, but what really matters is the power of iteration to amplify or dampen. The Butterfly Effect only makes sense within a frame of exponentiation.

However, one need not look to the future to argue for the importance of being able to notice and interpret transformations in exponential terms. We live in an age of perceptible exponentiation of many phenomena, including the planetary growth of human population, the increase of greenhouse gases in the atmosphere, the acidification of the ocean, the decline in species diversity, the increase in mechanical computational power, and the rising cost of housing. While some of these phenomena have been following a roughly exponential trajectory for a very long time, it is only recently that the transformations have accelerated to the point that we humans can literally see things happening before our eyes. In more mathematical terms, the slopes of the curves have become sufficiently steep to intersect with the time frames of human perception.

In a different vein, the emergent realization that many familiar phenomena follow a power-law distribution rather than, for example, a normal distribution is a very useful tool for understanding aspects of the world. Earthquakes, wars, income distribution, internet hubs, lunar craters, city sizes, solar flares, scaling laws in biological systems, fads and cultural trends, word frequencies in most languages, power outages—the list is an extensive one—obey a pattern of distribution in which the frequency of occurrence varies as a power of some obvious attribute (*e.g.,* intensity). The structural self-similarity of fractals offers another important example. In brief, for a phenomenon that follows a power-law distribution, there are few massive instances (*e.g.,* major quakes) and very many minor events (*e.g.,* minor tremors). As with all mathematical and statistical distributions, power-law distributions are approximations that fit some data sets better than others. However, the issue is less about the precision of the fit and more about the interpretive power of the principle.

The really important detail here is not that the capacity to interpret events and phenomena in exponential terms provides valuable means of noticing trends and understanding distributions. The real power is in the hope that it affords. Kurzweil's example of being only several doublings from energy sustainability is an especially cogent example. At least for myself, knowing that we are 8 units (in doubling terms) rather than 99 units (in ratio terms) from sustainability has prompted me to explore other energy sources in my own life. Whereas a factor-based interpretation left me feeling as though my efforts were irrelevant, a power-based reading alerts me to the importance of my own actions and decisions. A power-informed mindset not only better enables us to make sense of the world, it can empower us to engage and respond in fitting ways. It both signals threat and heralds promise.

Encountering exponentiation—personal and subpersonal

Appreciations of the processes and products of exponentiation are also useful for better understanding ourselves. For example, it turns out that many of our sensory systems operate in quasi-logarithmic ways. That is, they translate information that arrives across many levels of magnitude into scales that are experienced more as linear than exponential.

Perhaps the most familiar example is hearing. The decibel scale is a logarithmic tool that maps exponential increases in volume to our linear experiences of changing sound levels. Similarly, the smooth increase in pitch that we experience as a musical scale is played is actually precisely exponential. Each octave of rise corresponds to a doubling in frequency.

It also appears that sensitivities to touch, pain, and light that are experienced as smoothly linear are approximately logarithmic, which is why we can notice a mosquito lighting on skin and yet not suffer sensory overload when delivered a sharp blow. Vision is particularly impressive in its range of sensitivity: the human eye can pick up a 10^{10}-fold intensity variation.

Among those fields that study sensory capacities, these sorts of approximate logarithmic relationships between objective measures of stimulus strength and perception of that stimulus are instances of the Weber-Fechner Law, which offers that subjective sensation is proportional to the logarithm of the stimulus intensity for many of our perceptual systems, a biology-based predisposition that we seem to extend to our psychological readings of the world. In brief, we generally interpret whatever is experientially nearer as disproportionately (*i.e.,* literally not-multiplicatively, but exponentially) larger. A cover of *The New Yorker* from 29 March 1975 famously illustrates this point, as it critiques the habit of Manhattan residents of allowing the local to eclipse the world beyond. In the image, the distance between 9th and 10th Avenues is equivalent to the distance between 10th and the Hudson River, which matches the next interval to the Pacific Ocean, and the next interval, which encompasses the rest of the universe.

Such perceptual and psychological tendencies offer compelling reasons to study exponentiation: it is a useful construct for understanding how the world is experienced. A perhaps more compelling reason, however, is that the concept is already pervasive in school mathematics, although given comparatively little formal consideration. For the most part, the concept unfolds in a piecemeal way. The most obvious and persistent encounter is, of course, with the place-value numeration system, where the exponential ground of base-10 is only named after years of working with the symbols. And even when exponentiation is formally introduced, it is typically (and exclusively) defined as "repeated multiplication."

In fact, I have been unable to find any explicit elaborations of that initial definition in curriculum documents or classroom resources, anywhere, in spite of its obvious inadequacies when topics of fractional, irrational, negative, and zero exponents arise. I suspect that a principal reason that the standard definition is neither problematized nor elaborated has to do with the fact that almost all encounters with exponents are in application contexts—that is, in situations where a radius, a time interval, an intensity, or an unknown is repeatedly multiplied by itself. No need is presented to think in terms of other than the simplest of definitions.

That absence of other interpretations may be a shame, as it minimizes the need to really think through what exponentiation is all about, where it might be encountered, and how it might be a useful construct for better understanding the world and one's experience in the world. More significantly, I believe that more nuanced insights into the structure of the concept can support shifts in worldview. It is not just about better senses of the sizes of things (although that is vital); it is also about gaining insight into the shapes of things. To that end, I offer some thoughts on the pedagogy of exponentiation.

The what and why of exponentiation as a basic

In many ways, the topic of exponentiation, as it is currently represented in formal programs of study, textbooks, and many mathematics classrooms, is an exemplar of bad educational practice. As evidence of this assertion, consider the manner in which exponent rules are usually developed. Typical expectations are that students will gain facility with the following:

$$x^a \times x^b = x^{a+b}$$

$$x^a \div x^b = x^{a-b}$$

$$(x^a)^b = x^{ab}$$

$$(xy)^a = x^a y^a$$

$$(x/y)^a = x^a/y^a$$

$$x^{-a} = 1/x^a$$

$$x^0 = 1$$

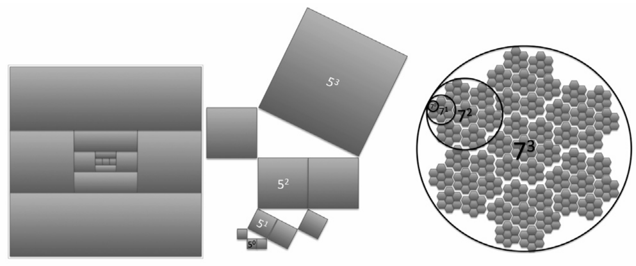

Figure 1. *Some images for illustrating exponential growth developed by students, showing powers of 3, 5, and 7, respectively (the figures for 5 and 7 were based on images found online).*

In my experience, each of these expressions is introduced, explained, and justified using a repeated-multiplication interpretation. For example, I have never encountered an explanation for why x^0 must equal 1 that does not first establish that $x^a \div x^b = x^{a-b}$ (through placing a count of a x's over a count of b x's, canceling, and noticing that "$a - b$" x's remain) and then setting a equal to b. That is, the justifications are offered entirely on the symbolic plane, pretty much ignoring advice from the last several decades to begin with bodily experiences, proceed by introducing symbolic means to describe and interpret those experiences, and only then move toward operating strictly on a symbolic level. Exponentiation and logarithms arrive in the reverse order: defined as an abstraction (of multiplication), dressed symbolically, and applied to events and problems (*e.g.,* repeated paper folding, the Rice-Doubling question, the Richter scale, half lives).

Consequently there is a poverty of interpretations for the concept of exponentiation, a point that was brought home to me in a recent presentation on an associated topic. I asked an audience of mathematicians and mathematics educators to identify instances and images of exponentiation. Many applications were identified, but when it came to visuals, the *only* image that was called out after several minutes of discussion was a curve showing compound interest. Of course this sophisticated audience was aware of many, many others and would have no doubt generated a rich list given time. But why was that richer list not at their fingertips?

In contrast, I recently used a similar prompt in an 8th-grade (aged 13-14 years) classroom in a weeklong inquiry into exponentiation. With more time and internet access, the resulting array of instances and images of exponentiation was much more extensive and considerably more varied, and included growth curves, explosions, logarithmic spirals, base-10 blocks, and some fractal images. (I omit most of the list here; engaging in the exercise of searching for and sorting through them is more valuable than a recounting of the outcomes.) As well, students created images of exponential growth and decay, oriented by an initial example that began with a single square on the whiteboard, then doubled, then the result doubled, and so on. Students were then invited to create similar images for values from 3 to 9 (see Figure 1 for some variations), with a further invitation to craft PowerPoint presentations for those who wished to work on the task outside of class time. The PowerPoint format allowed for dynamic illustrations of exponential change, a quality that is difficult to show in print format (see Figure 2).

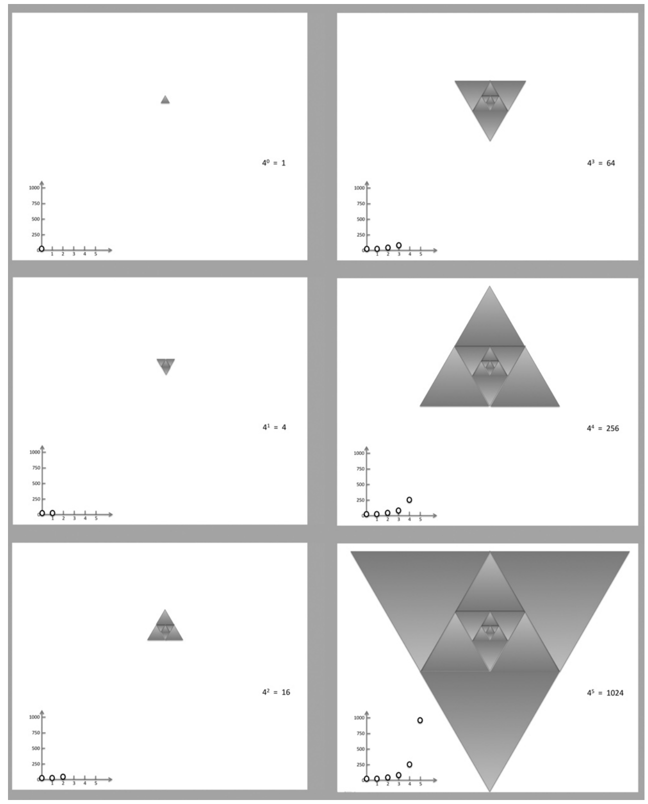

Figure 2. *A sequence extracted from an 8th-grade student's illustration of powers of 4. (The complete sequence also included negative exponents.)*

As powerful as the creation and discussion of these sorts of images was, however, the more impactful part of the inquiry for me began on our second day with a collective interrogation of the structure of mathematical invention, which was afforded by the connecting of exponentiation to addition and multiplication.

It bears mention that these students had been engaged in a sequence of "concept studies" (Davis & Renert, 2014) throughout the school year, and were thus familiar with deconstructing concepts. They were aware of varied instantiations of number (*e.g.*, as count, as measure, as position in space), addition (*e.g.,* as combining, as extending, as shifting), and multiplication (*e.g.,* as grouping, as scaling, as array/area making). They had also been introduced to exponentiation as repeated multiplication a few years prior, which appeared to be the only instantiation of the concept they could make explicit when the inquiry began.

Part of prior concept studies included building addition, subtraction, multiplication, and division "grids," which were variations on more familiar tables that are laid out on xy coordinate systems. I have found this device to be particularly powerful for interpreting, for example, identity elements, commutativity, and integer multiplication, and so thought it a reasonable starting place. A small portion of our exponentiation chart is presented in Figure 3 (overleaf). The actual chart spanned values of −10 to +10 on both axes (*i.e.,* in terms of absolute values represented on the grid, 10^{-10} to 10^{10}).

The grid was constructed in steps, starting with the first quadrant where there were no real surprises apart from a lack of symmetry along the $y = x$ line. (Owing to the commutativity of addition and multiplication, their grids are symmetric about $y = x$.) After working through examples to convince ourselves why that was the case, we moved to the fourth quadrant where students quickly noted and extended vertical patterns of decrease from the first quadrant. (The decision to record 1 as 1/1 in that quadrant was theirs.)

The left side of the grid occupied most of the session, as a number of quandaries arose that demanded attention. Predictably, the oscillation between positive and negative values was uncomfortable for many, but explanations were quickly offered. The more compelling question for most was around, "What happens between the rows?" on the left hand side. Their calculators could handle fractional exponents and positive non-integral bases (*e.g.,* $4^{2.6}$), but spat out ERROR when negative bases were used (*e.g.,* $(-4)^{2.6}$). I deflected the queries, advising that there were online resources for anyone wishing to delve into the emergent issues [6].

As part of this deflection, I did mention that a new number system is actually needed to talk about what lurks between the lines of the exponentiation grid. While I elected not to delve explicitly into imaginary and complex numbers, I drew an analogy with other operation charts and other number systems. In particular, the need for signed numbers arose in creating addition and subtraction grids, and for fractional numbers when creating multiplication and division grids. It made sense that another operation might present the need for another set of numbers.

Somewhat unexpectedly, the manner of deflection actually set the stage for the third and fourth of the five sessions, which were almost entirely focused on exploring analogies (or lack thereof) between exponentiation and prior operations. To kick-start discussions, I noted that the symbolism for exponentiation is problematic, as it obscures the relationship to other operations. To highlight similarities with "2 + 3" and "2 × 3", I proposed "2 ↑ 3" (read: "2 exponentiated by 3"). Some of the subsequent topics of speculation and discussion, all identified by the students, included:

- Just as addition and multiplication have inverse operations, there must be an inverse operation for exponentiation. It was proposed to be "de-exponentiation," signified by ↓ (*e.g.*, 8 ↓ 3 = 2, which is more conventionally written $8^{1/3} = 2$ or $\log_2 8 = 3$; I elected to mention logarithms at this point, but the topic was not pursued).
- 1 must be the exponential identity element because $a \uparrow 1 = a$ and $a \downarrow 1 = a$.
- Each number must have an exponentive inverse, "$\downarrow a$" (akin to additive inverse and multiplicative inverse), such that $a \uparrow (\downarrow a)$ = the identity.
- Just as subtracting can be rewritten as "adding the additive inverse" and dividing as "multiplying by the multiplicative inverse," de-exponentiation should be interpretable as "exponentiating by the exponentive inverse," *i.e.,* $a \downarrow b = a \uparrow (\downarrow b)$.

$x \uparrow y$	−5	−4	−3	−2	−1	0	+1	+2	+3	+4	+5
+5	−3125	−1024	−243	−32	−1	0	1	32	243	1024	3125
+4	+625	+256	+81	+16	+1	0	1	16	81	256	625
+3	−125	−64	−27	−8	−1	0	1	8	27	64	125
+2	+25	+16	+9	+4	+1	0	1	4	9	16	25
+1	−5	−4	−3	−2	−1	0	1	2	3	4	5
0	1	1	1	1	1	?	1	1	1	1	1
−1	$-\frac{1}{5}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{1}$	0	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
−2	$+\frac{1}{25}$	$+\frac{1}{16}$	$+\frac{1}{9}$	$+\frac{1}{4}$	$+\frac{1}{1}$	0	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$
−3	$-\frac{1}{125}$	$-\frac{1}{64}$	$-\frac{1}{27}$	$-\frac{1}{8}$	$-\frac{1}{1}$	0	$\frac{1}{1}$	$\frac{1}{8}$	$\frac{1}{27}$	$\frac{1}{64}$	$\frac{1}{125}$
−4	$+\frac{1}{625}$	$+\frac{1}{256}$	$+\frac{1}{81}$	$+\frac{1}{16}$	$+\frac{1}{1}$	0	$\frac{1}{1}$	$\frac{1}{16}$	$\frac{1}{81}$	$\frac{1}{256}$	$\frac{1}{625}$
−5	$-\frac{1}{3125}$	$-\frac{1}{1024}$	$-\frac{1}{243}$	$-\frac{1}{32}$	$-\frac{1}{1}$	0	$\frac{1}{1}$	$\frac{1}{32}$	$\frac{1}{243}$	$\frac{1}{1024}$	$\frac{1}{3125}$

Figure 3. A piece of our exponentiation table/grid.

- There must be a "next operation" of repeated exponentiation [7].

There were other speculations as well, but I highlight these ones to illustrate an important quality of secondary school mathematics. Whereas almost all the concepts encountered at the elementary level can be interpreted in terms of (*i.e.,* are analogical to) objects and actions in the physical world, the analogical correlates of concepts at the secondary level tend to be to mathematical objects and actions. While this entails a leap in abstraction, it corresponds to a leap in mathematical power (see Hofstadter & Sander, 2013). Explicit analogy, then, is both a mechanism for extending mathematical insight and a window into the structure of mathematics knowledge.

Such was the thinking behind my initial suggestion to the class to use a symbolism that foregrounded the analogies to addition and multiplication. On this count, it is worth recalling that mathematics pedagogy has played an important role in selecting and standardizing notations for all of mathematics, as most cogently illustrated by Robert Recorde's proposals of "=," "+," and other now-standard symbols in a text intended for students [8].

The fifth and final session with the students was devoted to review and consolidation. Recalling my earlier assertion that quantity and form tend to be tightly coupled across domains of mathematical inquiry, I elected to frame the session by developing the table presented in Figure 4, through which I suggested that the geometric image best fitted to addition is the line, to multiplication is a rectangle, and to exponentiation is a fractal.

The balance of the session was given to looking across instances of exponential growth and decay (*e.g.,* population growth, species decline, greenhouse gas increase, technology evolution), framed by Charles and Ray Eames' (1977) film, *Powers of Ten* [9], and Cary and Michael Huang's (2012) interactive Prezi, *The Scale of the Universe* [10]. The students were, to put it mildly, highly engaged and, to my reading, in a strongly mathematical way. There were many and diverse indications of appreciations of the nature of the processes of increase and decline at work.

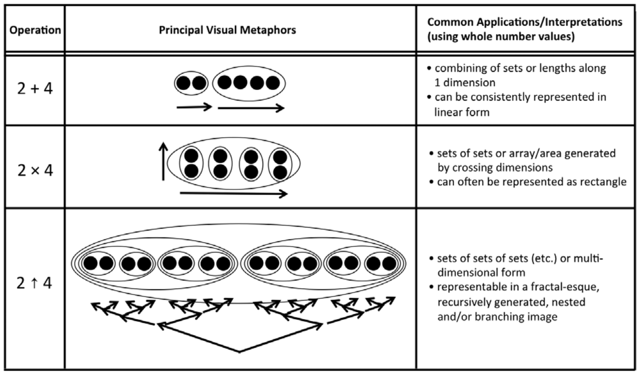

Figure 4. Some geometric analogies to arithmetic operations.

Never one to miss an opportunity to tie the mathematics to the grander experiences of learning and living, my capping comment on the week was to suggest that the images used to frame schooling and to structure curriculum were predominantly lines and rectangles … and that given the many and diverse trajectories represented in that room alone, perhaps the images of exponentiation were more fitting.

And so …?

As scholars of educational change are wont to note, sustainable transformation demands an approach that spans all levels of the system, and that is no simple task. The matter of taking up exponentiation as a new basic, then, is at this point more a musing than a proposal. The idea would have to be part of much more systematic and systemic reviews of the intents and contents of school mathematics. That said, I do feel the suggestion has merit as a case study to frame some of the more important considerations. For instance, at the societal level, exponentiation has emerged as a vital interpretive competence as the speed of cultural evolution has accelerated into the temporal space of personal experience. The topic also opens up a new set of visual metaphors for change and growth, as it reveals a deeply rooted and tacit reliance on lines and rectangles to organize and interpret the world (see Davis & Sumara, 2005, for a more complete discussion).

At the level of mathematics education, the topic of exponentiation presents a fresh opportunity to rethink structures of curriculum and pedagogy, unburdened by the centuries-deep, calculation-driven baggage of other basic operations [11]. In particular, the topic can be exploited as an entry point into the analogical character of mathematical insight. As I have attempted to illustrate in this article, exponentiation presents windows into both analogical-associative aspects of mathematics knowledge and the analogical-associative aspects of mathematics learning.

Further to the matter of individual learning, it is worth noting that none of the students in the above episode wondered aloud why they were studying the topic or engaging in the exercises. Image rich, example dense, and rife with mathematical connections—which is to say, not subject to the impoverishment of routinized activity—there was no need to invest effort in making it relevant.

Notes

[1] The symposium was held on March 16, 2013, as part of the Research Pre-session of the Annual Meeting of the National Council of Teachers of Mathematics in Denver, CO. It was entitled, "'There's an app for that,' but how good is it?" and was led by U. Kotelawala, L. M. Gellert, K. Offenholley, and R. J. Graham.

[2] en.wikipedia.org/wiki/Mathematics, accessed 15 April 2013.

[3] www.ted.com/talks/ray_kurzweil_on_how_technology_will_transform_us.html

[4] The talk was posted in November 2006. Presumably the number of doublings has declined.

[5] From www.ted.com/talks/ray_kurzweil_on_how_technology_will_transform_us.html The quotation can be found via the "Show Transcript" button.

[6] Several students did take up this invitation, and located an online calculator that handles complex numbers at www.mathisfun.com.

[7] There is: tetration. (In terms of the grid in Figure 3, just as the $x = y$ diagonal on the addition chart corresponds to the $y = 2$ line on the multiplication chart, and the $x = y$ diagonal on the multiplication chart corresponds to the $y = 2$ line on the exponentiation chart, so does the $x = y$ diagonal on the exponentiation chart correspond to the $y = 2$ line on the tetration chart.)

[8] While it might appear that there is consensus on notation for exponentiation, this is not the case. Current variations include a^b, $a\hat{}b$, $a{**}b$, and $a\uparrow b$. Similarly, there is variation on how symbols might be read, with options including"*a* raised to the power of *b*," "*a* to the *b*," and "the *b*th power of *a*." All of these popular options obscure the analogical relationships between exponentiation and its prior operations.

[9] The film can be viewed at www.powers of10.com. The site includes several newer interactive features.

[10] Available at htwins.net/scale2/

[11] Note that I in no way mean to suggest that calculation is a troublesome aspect of school mathematics. On the contrary, in the weeklong episode reported in this writing, roughly half the time involved calculations of one sort or another—but this work was never undertaken for its own sake. Rather, calculation was always in the service of developing conceptual understanding.

References

Davis, B. (1996) *Teaching Mathematics: Toward a Sound Alternative.* New York, NY: Routledge.

Davis, B. & Renert, M. (2014) *The Math Teachers Know: Profound Understanding of Emergent Mathematics.* New York, NY: Routledge.

Davis, B. & Sumara, D. (2005) Challenging images of knowing: complexity science and educational research. *International Journal of Qualitative Studies in Education* **18**(3), 305–321.

Grumet, M. R. (1995) The curriculum: what are the basics and are we teaching them? In Kincheloe, J. L. & Steinberg, S. R. (Eds.) *Thirteen Questions: Reframing Education's Conversation* (2nd edition), pp. 15–21. New York, NY: Peter Lang.

Hofstadter, D. & Sander, E. (2013) *Surfaces and Essences: Analogy as the Fuel and Fire of Thinking.* New York, NY: Basic Books.

Kurzweil, R. (2005) *The Singularity is Near: When Humans Transcend Biology.* New York, NY: Viking.

NCTM (2006) *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics: A Quest for Coherence.* Reston, VA: NCTM.

Wagner, D. & Davis, B. (2010) Feeling number: grounding number sense in a sense of quantity. *Educational Studies in Mathematics* **74**(1), 39–51.