SPREADSHEETS AS A TRANSPARENT RESOURCE FOR LEARNING THE MATHEMATICS OF ANNUITIES

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Mathematicians compress, mathematics teachers decompress. This statement summarises a key insight that is gaining widespread acceptance in the research literature. Mathematicians strive to compress ideas, processes and relationships into concise and abstract symbolic forms, thus increasing efficiency and usability. However, it is these reifications that often become barriers to learning mathematics, particularly at school. So mathematics teachers need to decompress or unpack the mathematics to make it accessible for pupils. Ball, Bass and Hill (2004) argue that unpacking is one of the distinctive features of mathematical knowledge for teaching and that an ability to unpack mathematics reflects mathematical knowledge that is usable for the work of teaching (Ball & Bass, 2000).

In this article I share a story from a financial maths course for teachers where two student teachers use spreadsheets to make sense of annuities. I argue that the transparency of the spreadsheet (Lave & Wenger, 1991) enables the students both to unpack the mathematical and financial ideas of annuities and to translate between symbolic forms (Davis & Simmt, 2006). In unpacking and translating, the students display mathematical knowledge that appears to be usable for teaching.

Background

In South Africa the mathematical preparation of pre-service secondary teachers is generally located in schools of education which makes the South African situation relatively unique. In our programme we offer a 12-week course in financial mathematics, a topic that has been given increased attention in the new school mathematics curriculum. In this course students engage with introductory financial mathematics, aspects of the world of finance and economics, and teaching. This multiple focus distinguishes the course from introductory financial maths courses offered to commerce and economic science students in the University. A key goal is that the applied nature of the course should enable students to develop a deeper understanding of the world of finance, of mathematics and of the relationships between the two. We believe that the depth of understanding that is gained through such contextualisation of mathematics is central to preparing future teachers to teach financial maths in schools, and in community contexts more broadly.

The data discussed here are drawn from a larger study investigating the mathematical learning of pre-service secondary teachers in the financial mathematics course where I am the teacher researcher. [1] Prior to the episode discussed below, students had been required to model a scenario representing the future value of an ordinary annuity [2], which included deriving an appropriate formula. They had also modelled an annuity due situation involving missing a monthly payment and trying to make up for this by doubling the following month's payment. In this work students made use of the spreadsheet shown in figure 1, which I had introduced to them.

The spreadsheet represents a scenario of 12 monthly payments of R250 at 6% p.a. compounded monthly. The payments are made at the end of each month, and at the end of December the total amount accumulated is R3083.89. Each row represents a monthly payment from January to December, and each 250 represents the value of the deposit at the time it is made. Thereafter each payment grows at 0.5% per month. Each row thus constitutes a geometric progression. The balance in the account at the end of any month can be calculated by summing that column. For example the value in cell F22 (R1262.56) represents the balance in the account at the end of May. The values in the columns form the same geometric progression as the rows, and each entry in a column indicates the value that a particular payment is contributing to the account balance at a given month end. This numeric representation, and the associated process for calculating the future value of the annuity can be compressed into the algebraic formula presented in figure 2.

My concern throughout the course was that the student teachers should be able to unpack or decompress formulae like the one above, and re-examine how any particular payment impacts the account balance. Frequently once annuity formulae have been derived, they become new objects whose use is characterised by substitution of values to solve for an unknown. As a result the formulae become black boxes that hide the ways in which money grows over time. While the formulae are important for efficient calculation, their structures provide little insight into the ways in which regular payments each gain interest over time, and how these accumulate. The spreadsheet, on the other hand, opens up the inner working of the annuity calculation enabling one to focus on the account balance and on individual payments. For me this had always been one of the main values of the spreadsheets for my own and students' understanding, and I knew that students recognised the value of the spreadsheet. But a surprise was waiting for me …

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Investments with monthly payments at end of period**												
2													
3	**Monthly Payment**	250											
4	**Rate**	6	% p.a. compounded monthly										
5													
6		End of month											
7		**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
8		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
9		**250.00**	251.25	252.51	253.77	255.04	256.31	257.59	258.88	260.18	261.48	262.79	264.10
10			**250.00**	251.25	252.51	253.77	255.04	256.31	257.59	258.88	260.18	261.48	262.79
11				**250.00**	251.25	252.51	253.77	255.04	256.31	257.59	258.88	260.18	261.48
12					**250.00**	251.25	252.51	253.77	255.04	256.31	257.59	258.88	260.18
13						**250.00**	251.25	252.51	253.77	255.04	256.31	257.59	258.88
14							**250.00**	251.25	252.51	253.77	255.04	256.31	257.59
15								**250.00**	251.25	252.51	253.77	255.04	256.31
16									**250.00**	251.25	252.51	253.77	255.04
17										**250.00**	251.25	252.51	253.77
18											**250.00**	251.25	252.51
19												**250.00**	251.25
20													**250.00**
22	**Account balance**	**250.00**	**501.25**	**753.76**	**1007.53**	**1262.56**	**1518.88**	**1776.47**	**2035.35**	**2295.53**	**2557.01**	**2819.79**	**3083.89**

Figure 1. Spreadsheet showing individual payments

Using future value to think about present value of an annuity

We had moved on to model present value of annuity situations, such as monthly payouts from a lump sum, or loan repayments. The class was asked to investigate a scenario where a student wanted to save enough money from her December vacation work to make monthly withdrawals of a given amount throughout the following academic year (which is February to November in South Africa). The class came up with four different formulae that could model the situation appropriately. One of the formulae closely resembled the outstanding balance formula as shown in figure 3.

The class was not yet aware that it was the outstanding balance formula and for this reason I will call it the *JS formula* after the students who produced it. In the following session students worked in their tutorial groups to explore the four formulae, deciding which ones were appropriate models, what various symbols represented, and possible links between formulae.

Hailey, Sakhile, Lena and Jefferson were working together and quickly recognised that three of the formulae were structurally equivalent, and were all instances of the typical present value of an annuity formula with minor differences in notation. They realised that the JS formula worked but did not understand why it worked nor where it came from. Hailey then saw a link between the spreadsheet in figure1 and the JS formula, and proceeded to adapt the spreadsheet as shown figure 4. The spreadsheet represents the following situation: a student wants to make monthly withdrawals of R250 from February to November, at the end of every month. How much must she invest at the end of January if interest is 6.75% compounded monthly?

Hailey took the spreadsheet which consisted of an "upper triangle" only, and completed it to become a rectangle. She also adjusted it to deal with the relevant months. To determine the values for the "lower triangle" she depreciated each payment until January. Thus each R250 withdrawal is moved back in time to the end of January. So in her spreadsheet there are 10 rows representing monthly withdrawals from February to November. The different values in each row come from the same geometric progression but have different starting values which are determined by the position of the 250 in the sequence. The different positions of the 250 indicate the student is withdrawing R250 at the end of each month. In moving each of the 10 payments back in time to the end of January, one is able to calculate the value of the lump sum that must be deposited (R2424.37) to enable 10 withdrawals of R250 (with a zero closing balance).

The spreadsheet shows 11 columns of values. In summing the January column, one is able to calculate the lump sum that must be deposited at the end of January. In the remaining columns, the totals give the balance in the account at the end of each month, assuming no withdrawals or deposits are made. If one adds only the values in each column that lie *below* the 250, they will give the outstanding

$FV = pymt\left[\frac{(1+i)^n - 1}{i}\right]$	… where: FV = the accumulated (or future) value $pymt$ = monthly payment i = monthly interest rate n = number of payments

Figure 2. Formula for future value of an ordinary annuity

$F_v = P_v(1+i)^n - x\left[\frac{(1+i)^n - 1}{i}\right]$	… where: P_v = amount saved by student, x = monthly withdrawals, i = monthly interest rate, n = no. of withdrawals, F_v = 0 since the goal was to withdraw all the savings

Figure 3. JS formula for present value of annuity

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Investments with monthly payments at the end of period**												
2													
3	**Monthly payment**		250										
4		**Rate**	6.75	% p.a. compounded monthly									
5													
6		End of Month											
7		**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
8		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
9		248.60	**250.00**	251.41	252.82	254.24	255.67	257.11	258.56	260.01	261.47	262.94	
10		247.21	248.60	**250.00**	251.41	252.82	254.24	255.67	257.11	258.56	260.01	261.47	
11		245.83	247.21	248.60	**250.00**	251.41	252.82	254.24	255.67	257.11	258.56	260.01	
12		244.45	245.83	247.21	248.60	**250.00**	251.41	252.82	254.24	255.67	257.11	258.56	
13		243.09	244.45	245.83	247.21	248.60	**250.00**	251.41	252.82	254.24	255.67	257.11	
14		241.73	243.09	244.45	245.83	247.21	248.60	**250.00**	251.41	252.82	254.24	255.67	
15		240.37	241.73	243.09	244.45	245.83	247.21	248.60	**250.00**	251.41	252.82	254.24	
16		239.03	240.37	241.73	243.09	244.45	245.83	247.21	248.60	**250.00**	251.41	252.82	
17		237.69	239.03	240.37	241.73	243.09	244.45	245.83	247.21	248.60	**250.00**	251.41	
18		236.36	237.69	239.03	240.37	241.73	243.09	244.45	245.83	247.21	248.60	**250.00**	
19													
20													
21	**Balance**	2424.37	2438.00	2451.72	2465.51	2479.38	2493.32	2507.35	2521.45	2535.63	2549.90	2564.24	0.00
22													

Figure 4. Hailey's adapted spreadsheet

balance in the account at the end of each month (once the R250 has been withdrawn).

It is not clear whether Hailey was aware of all these relationships when she first developed the spreadsheet. For her it was not the numbers in the spreadsheet that were important but the geometric figures that she could see and how these related to the JS formula. The rectangle represented the lump sum of R2424.37 being invested at the end of January and receiving interest until the end of November. This made it easy for her to account for the first part of the JS formula ($P_v(1 + i)^n$) because it represented the initial amount growing without any withdrawals or deposits. Now she needed to explain why in a situation dealing with *present* value of an annuity, the *future* value formula was being used. She produced the following explanation.

The lower triangle represents money that was gaining interest in the account before being withdrawn, the upper triangle represents the money that is no longer in the account because the R250 has been withdrawn. So while the upper triangle shows each R250 payment accumulating beyond the R250 value, this is the amount that must be subtracted because it is no longer in the account and so cannot earn interest. For Hailey this explained why the entire upper triangle which represented the future value of the annuity must be subtracted.

Spatially she was seeing a rectangle and the upper triangle, and she was subtracting the triangle from the rectangle (see fig. 5). The lower triangle represents the present value of an annuity but the outstanding balance formula does not foreground this. So while the other three formulae represented only the lower triangle, the JS formula represented "rectangle minus triangle" and Hailey had found a way of explaining why it worked.

When she explained her discovery to the group. Jefferson and Lena accepted it readily although it became clear later that there were aspects which they had not fully understood. Sakhile was far less willing to accept it at first but after grappling with it over several days, he became an enthusiastic convert, and in an interview six months after the course had finished, he was able to draw on this representation to solve a problem involving outstanding balance.

Unpacking, translating and transparency

Ball *et al.* (2004) argue that one of the key components of teachers' mathematical work is that of unpacking – opening up mathematical ideas and mathematical objects for deeper scrutiny so that students can gain a fuller understanding of them. It seems obvious that unless a teacher can first unpack mathematical ideas for herself, she will be unable to do it for others. Linked to this, Davis and Simmt (2006) argue that a key competence for teachers is an ability to "*translate* notions from one symbolic system to another" (p. 303) such as working between various representations, images, metaphors and analogies. Hailey and Sakhile both demonstrated an ability to unpack and translate ideas about annuities. Their ability to do this gave them deeper insight into the mathematics and the financial issues at work. The spreadsheet representation that enables one to track the inner workings of particular payments and thus to see how the individual parts make up the whole. This can be done for an individual payment or for the account balance at any particular month end. The student teachers went further, seeing

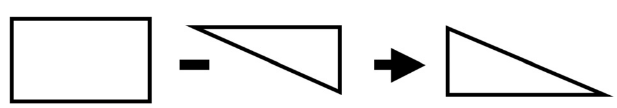

Figure 5. Hailey's geometric interpretation of the spreadsheet

beyond the numbers in the spreadsheet to new spatial images. Following Adler (1999, 2000), Ainley (2000) and Meira (1998) I draw on Lave and Wenger's (1991) notion of transparency of resources is useful in discussing what is happening here, and I would argue that the student teachers' ability to unpack and translate is dependent on the transparency of the spreadsheet for them.

In the context of mathematics, transparency is concerned with the ways in which students' use of resources enables them to gain access to the knowledge, processes and practices of mathematics. It is not an inherent characteristic of a resource. Rather a resource becomes transparent for a user through its use for a particular purpose in a particular setting. The transparency of a resource emerges through its dual characteristics of visibility and invisibility: it must be visible so that it can be seen and used, but must also be invisible so that through its use, it supports access to mathematics. When learners first work with new technologies such as spreadsheets, they often struggle to see the mathematics they are doing because the new technology is "in their face" – it is the object of their attention, too much in the foreground and hence too visible to provide access to the mathematics. Once they become more familiar with the technology, they are able to attend more to the mathematical ideas. The tool recedes into the background and thus becomes invisible. In order to learn about new functions of the technology and to work with it in new ways, the tool will become more visible again but will later recede once more into the background.

In the case of the spreadsheet, it's visibility lies in the numbers and the relationships between the numbers as they are hard-coded into the spreadsheet. As the student teachers worked with this, they were able to overlay geometric images on the spreadsheet which enabled them to collapse ranges of values into two different processes – the process of compounding a single deposit and the process of an annuity-based investment. Already at this stage they were seeing through the spreadsheet to both the mathematical structures contained in the JS formula and to the financial context of the problem – hence the invisibility of the spreadsheet.

The invisibility of a resource also lies in the ability to foreground and background (possibly even ignore) certain features. There are at least two instances where this occurs. Firstly, in seeing through the spreadsheet, they had to give a double meaning to each 250. It was simultaneously the amount that is withdrawn and the amount that grows but which isn't there. In this sense the 250's belong to the upper *and* lower triangles. This required them to interpret the spreadsheet even beyond the rectangle and triangle. Secondly, the final row of the spreadsheet gives the account balance. If one were strictly modelling the remaining balance in the account, then the values in this row should be diminishing to zero. In the adapted spreadsheet the summing of columns enabled students to assign meaning to the first part of the JS formula but this no longer produced the correct values for the account balance. They ignored this fact because it did not fit with the way they wanted to interpret the representation.

Conclusion

Hailey used the spreadsheet to unpack the mathematical situation her group was wrestling with. This was done to the level of individual payments which were then compressed to the level of particular investments – either a lump sum or an annuity. The recompression led to shifts in representation – from numeric to geometric and ultimately back to algebraic – which was the initial stimulus for the work. I would argue that the students' ownership and manipulation of the spreadsheet, their insight into the embedded relationships, and their ability to transcend the numeric representation are examples of productive and promising ways in which (future) teachers can learn (and/or relearn) mathematics in ways that are useful for the work of teaching. The students displayed a knowledge of annuities that moves between compressed formulae, elaborated numerical relationships in the spreadsheet and compressed representations of the spreadsheet. This reflects an ability to unpack mathematics and repack it into a new symbolic form. It is not restricted to algebraic formulae nor is it dependent on a multiplicity of values in a spreadsheet.

Notes

[1] This research is based on work supported financially by the Thuthuka programme of the National Research Foundation. Any opinions, findings and conclusions or recommendations expressed are those of the author and therefore the NRF does not accept any liability in regard thereto. The development of the Financial Mathematics course was supported by the Hermann Ohlthaver Trust.

[2] An ordinary annuity situation is one where the regular payments are made at the end of the period, whereas an annuity due is one where payments are made at the beginning of the period.

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